# Differential Equations and Calculus

## From Scratch!

$$\ddot{x} = -kx$$

**Matthias Müller**
**Ten Minute Physics**

# Change

- Physics is about change
- Forces change velocities
- Velocities describe the change of positions
- No change, no physics

by Frits Ahlefeldt

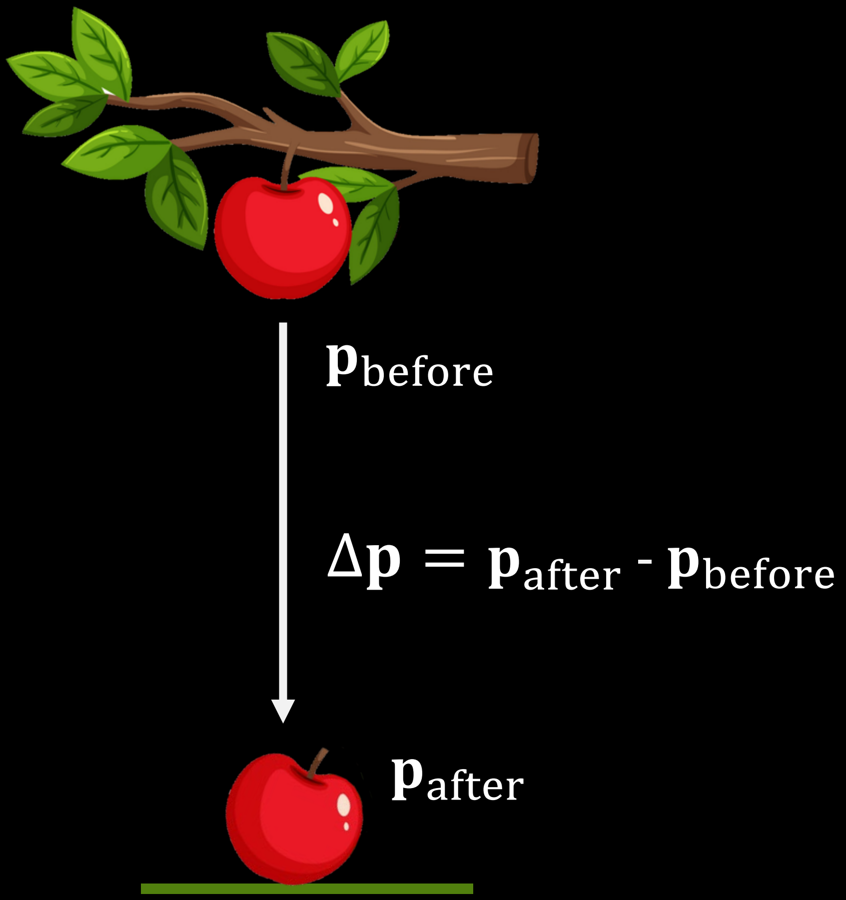
# Measuring Change

$$\Delta x = x_{\text{after}} - x_{\text{before}}$$

↑

Greek delta for difference

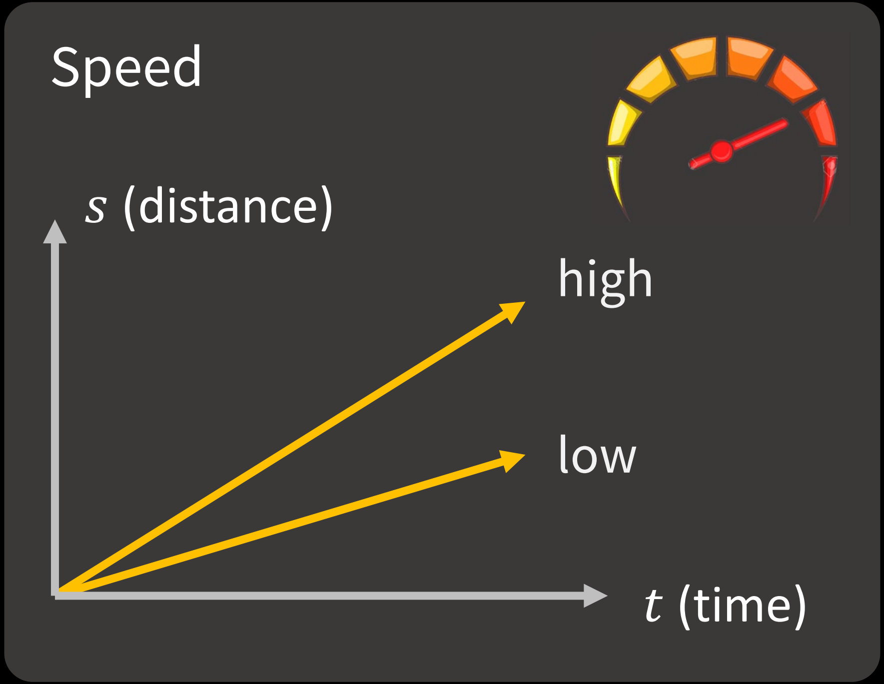
From 1pm to 5pm: 5 hours – 1 hours = 4 hours



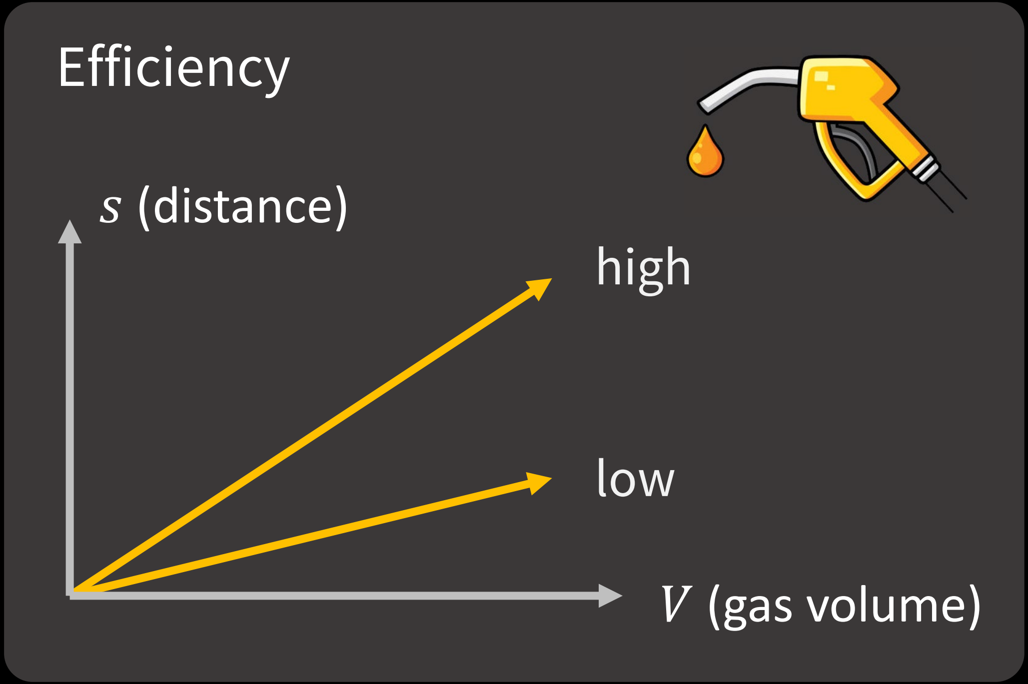
$$\Delta\mathbf{p} = \mathbf{p}_{\text{after}} - \mathbf{p}_{\text{before}}$$

# The Rate of Change

- The change with respect to another quantity (most often time)



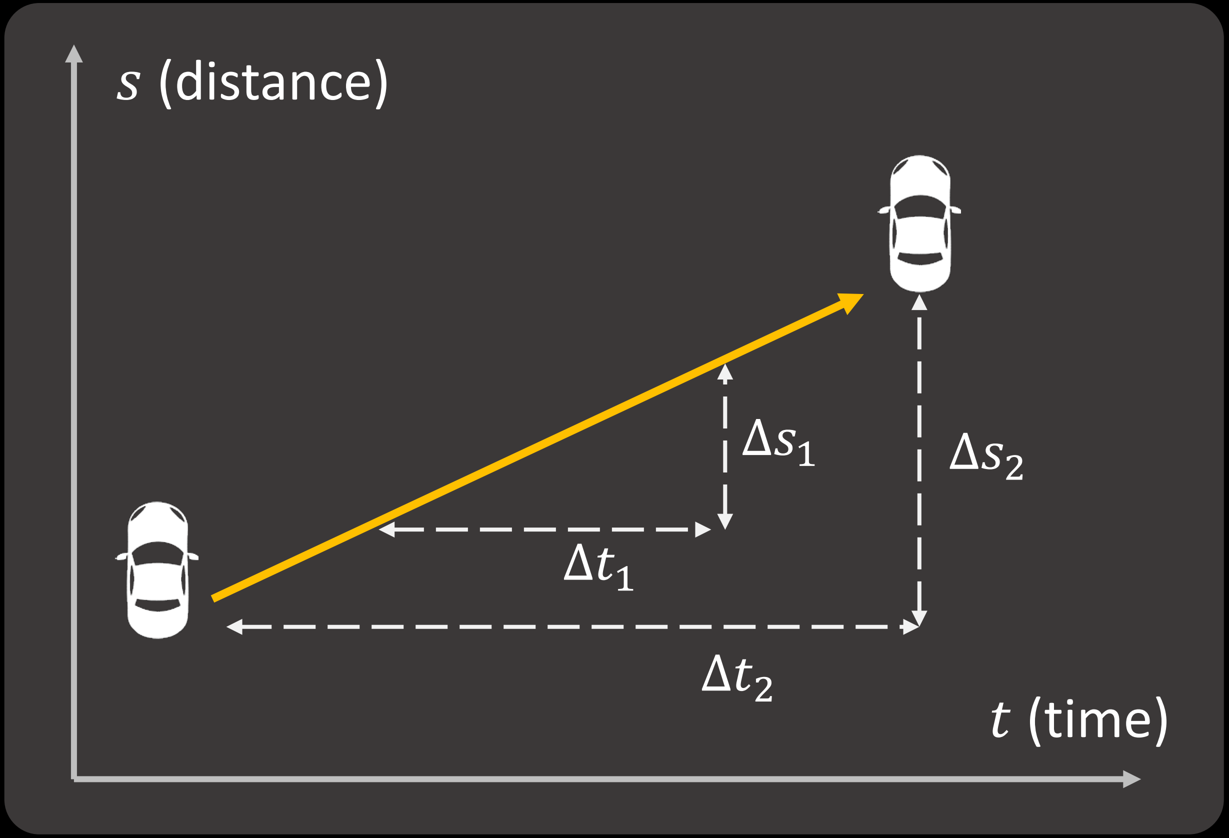
Speed



Efficiency

→ the rate of change is proportional to the slope!

# Measuring the Rate of Change

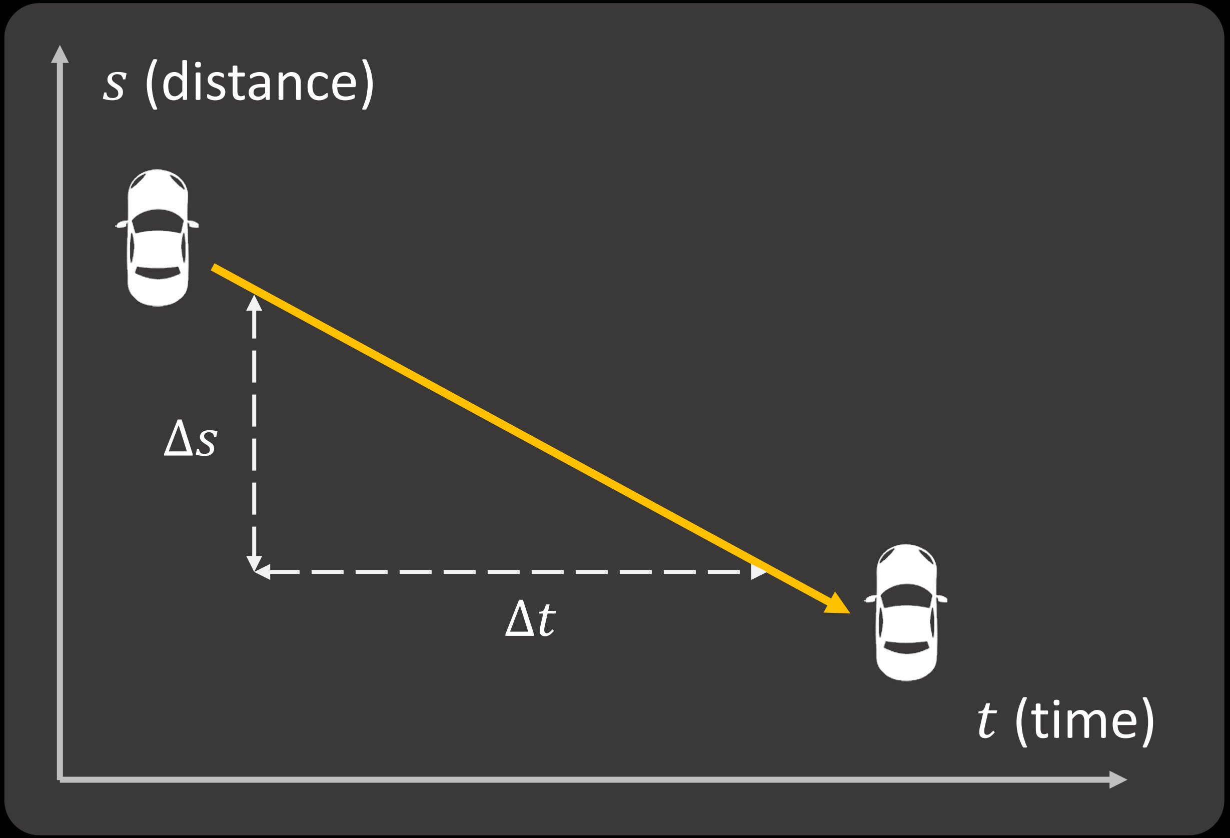


| | |
|---|---|
| $\Delta t$ | change in time |
| $\Delta s$ | change in position |
| $\frac{\Delta s}{\Delta t}$ | rate of change |

- constant for uniform change:

$$\frac{50\ km}{0.5\ h} = \frac{200\ km}{2\ h} = 100\,\frac{km}{h}$$

# Negative Rate of Change



$s_2 < s_1$

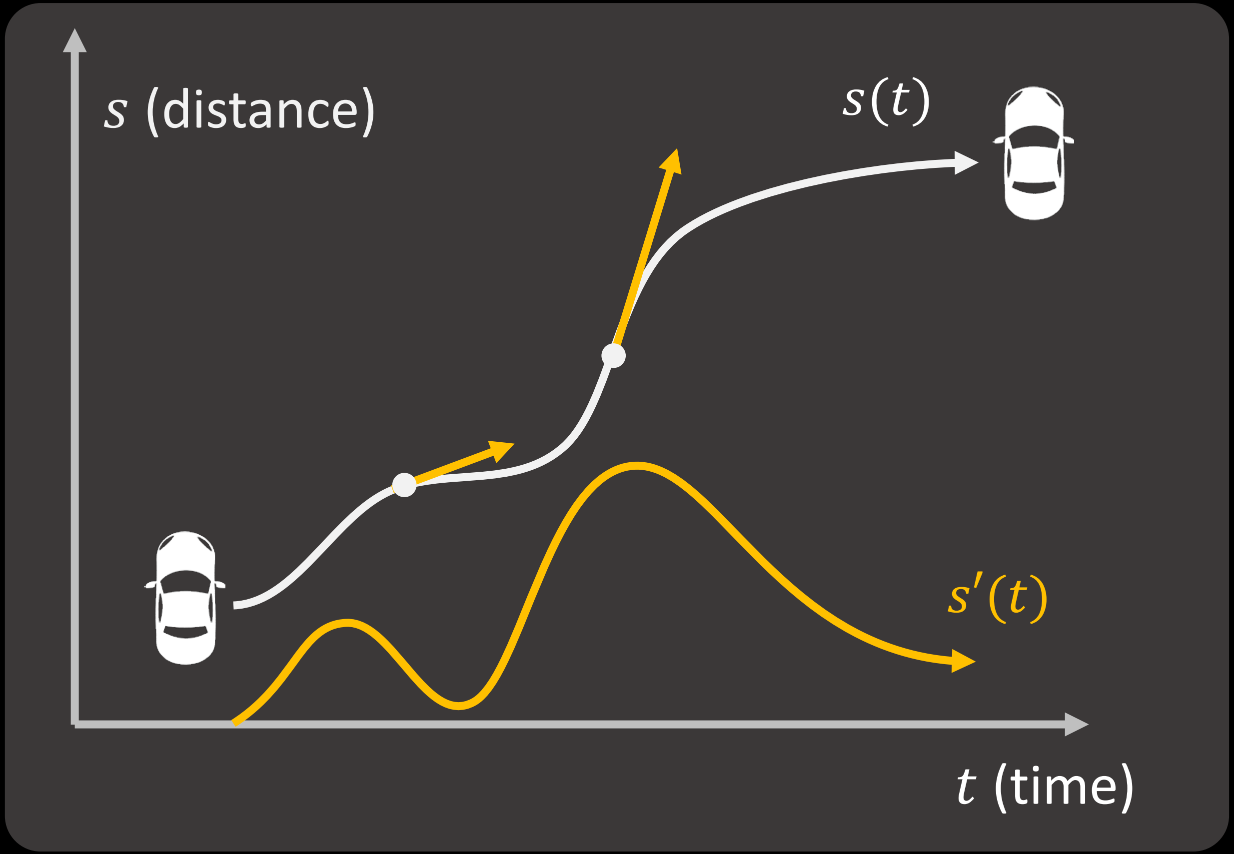
$\rightarrow \Delta s = s_2 - s_1 < 0$

$\rightarrow \dfrac{\Delta s}{\Delta \mathrm{t}} < 0$

→ a decrease

What if the rate of change
is not constant?

# Changing Rate of Change



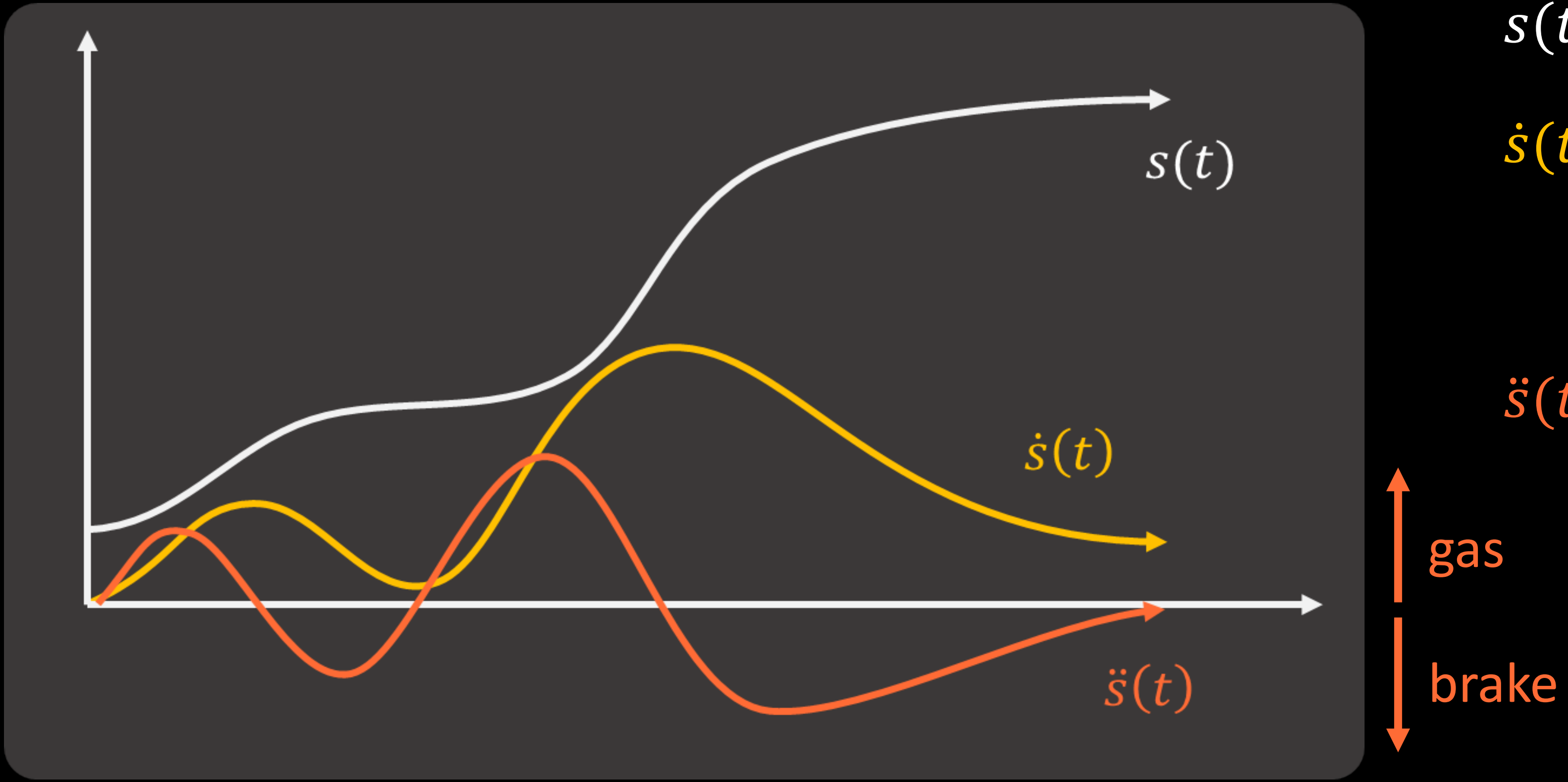
| | |
|---|---|
| $s(t)$ | function describing the position $s$ at time $t$ |
| s’$(t)$ | function describing the rate of change of $s(t)$ at time $t$ |

s’$(t)$ is called derivative of s$(t)$

velocity $v(t)$ in this case

$\dot{s}(t)$ for derivative w.r.t. time

# The Rate of Change of the Rate of Change



| | |
|---|---|
| $s(t)$ | position at time $t$ |
| $\dot{s}(t)$ | slope of $s(t)$<br>first derivative of $s(t)$<br>velocity at time $t$ |
| $\ddot{s}(t)$ | slope of $\dot{s}(t)$<br>curvature of $s(t)$<br>second derivative of $s(t)$<br>acceleration at time $t$ |

# Differential Equations

- Equations with functions and their derivatives as unknowns
- Required to describe physics mathematically!
- The basis for all physics simulations!

…to the best of my knowledge

# The Rabbit Equation

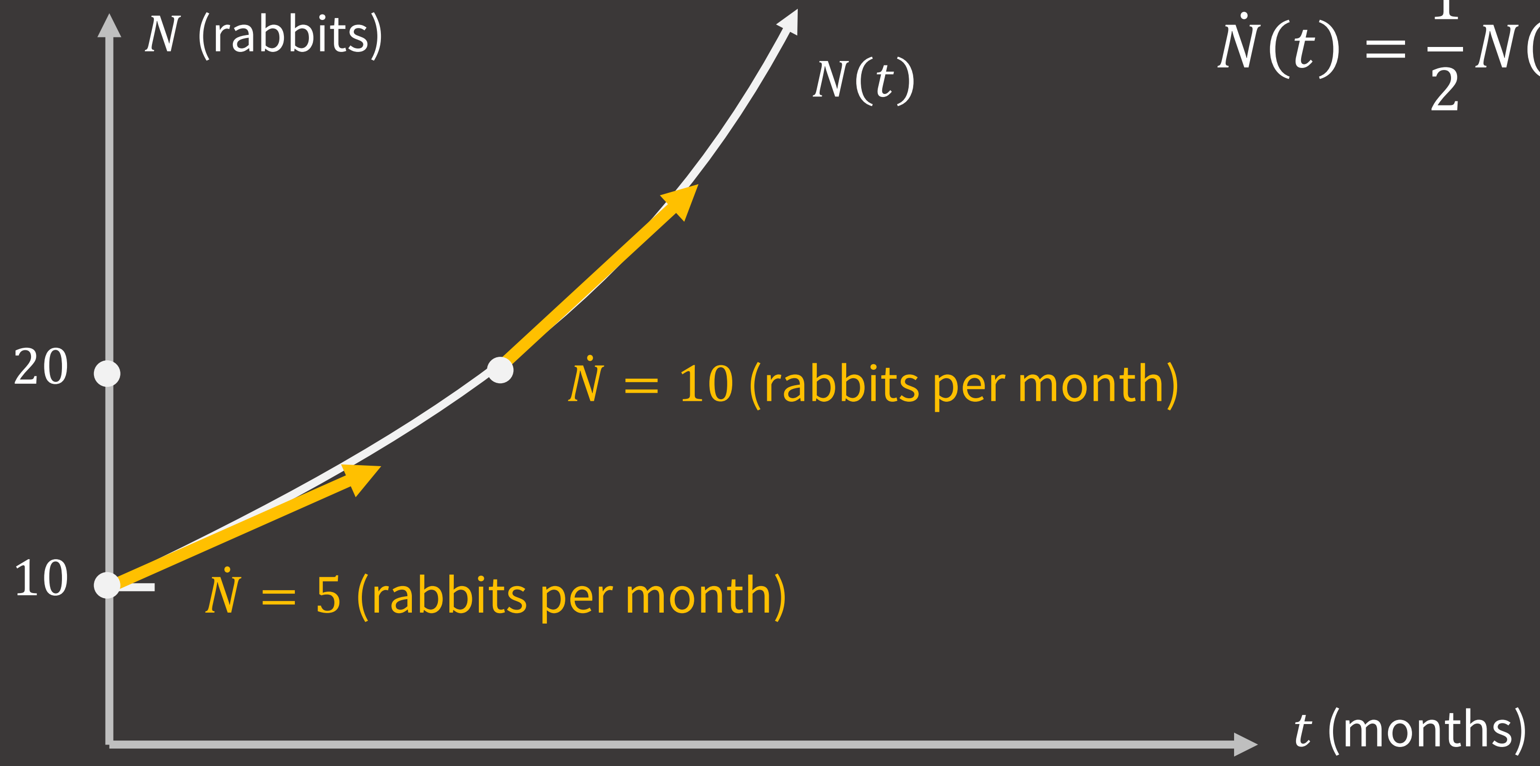
- $N(t)$ the number of rabbits at time $t$

$$\dot{N}(t) = k \cdot N(t)$$

- The rate of increase of the number of rabbits is proportional to the number of rabbits

- The more rabbits the more births

# Guessing the Solution



$$\dot{N}(t) = \frac{1}{2}N(t)$$

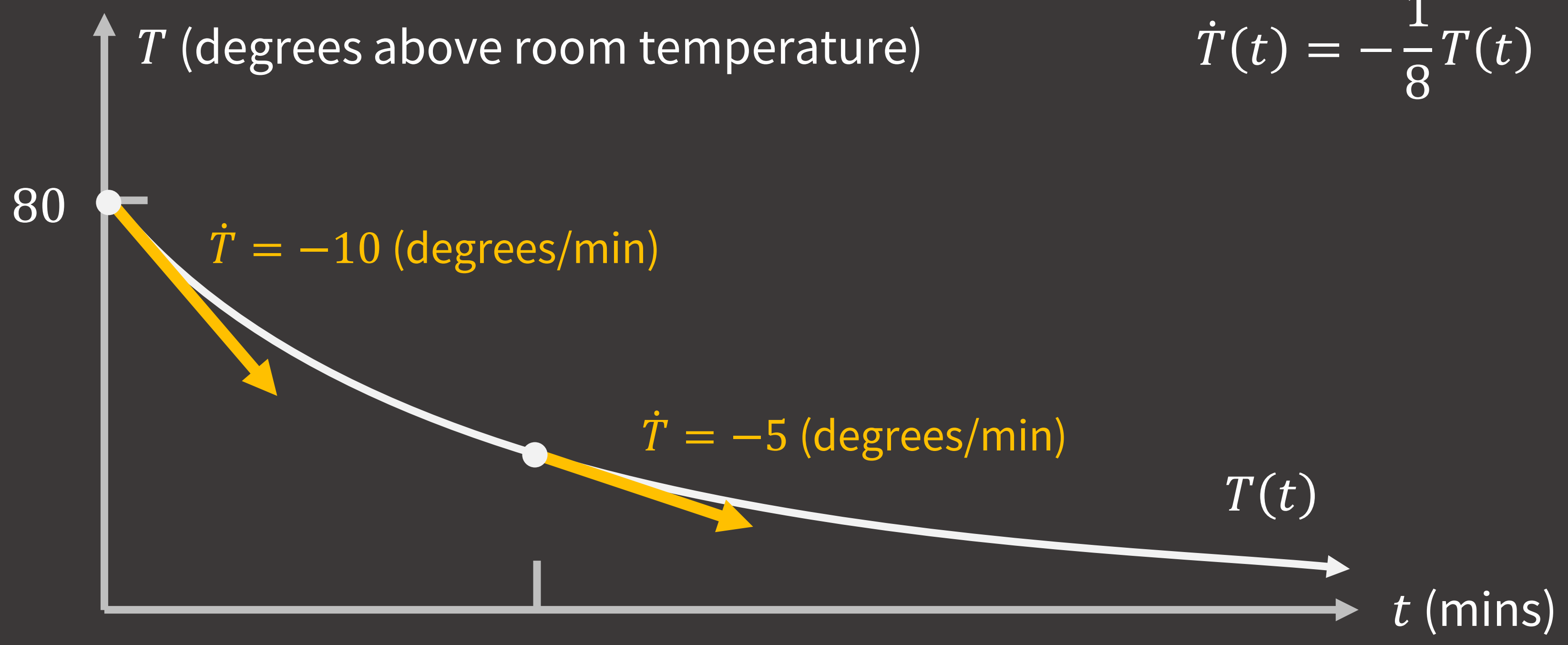
# The Coffee Equation

- $T(t)$ the temperature at time $t$

$$\dot{T}(t) = -k \cdot T(t)$$

- The rate of decrease of the temperature is proportional to the temperature
- The hotter the faster it cools down

# Guessing the Solution



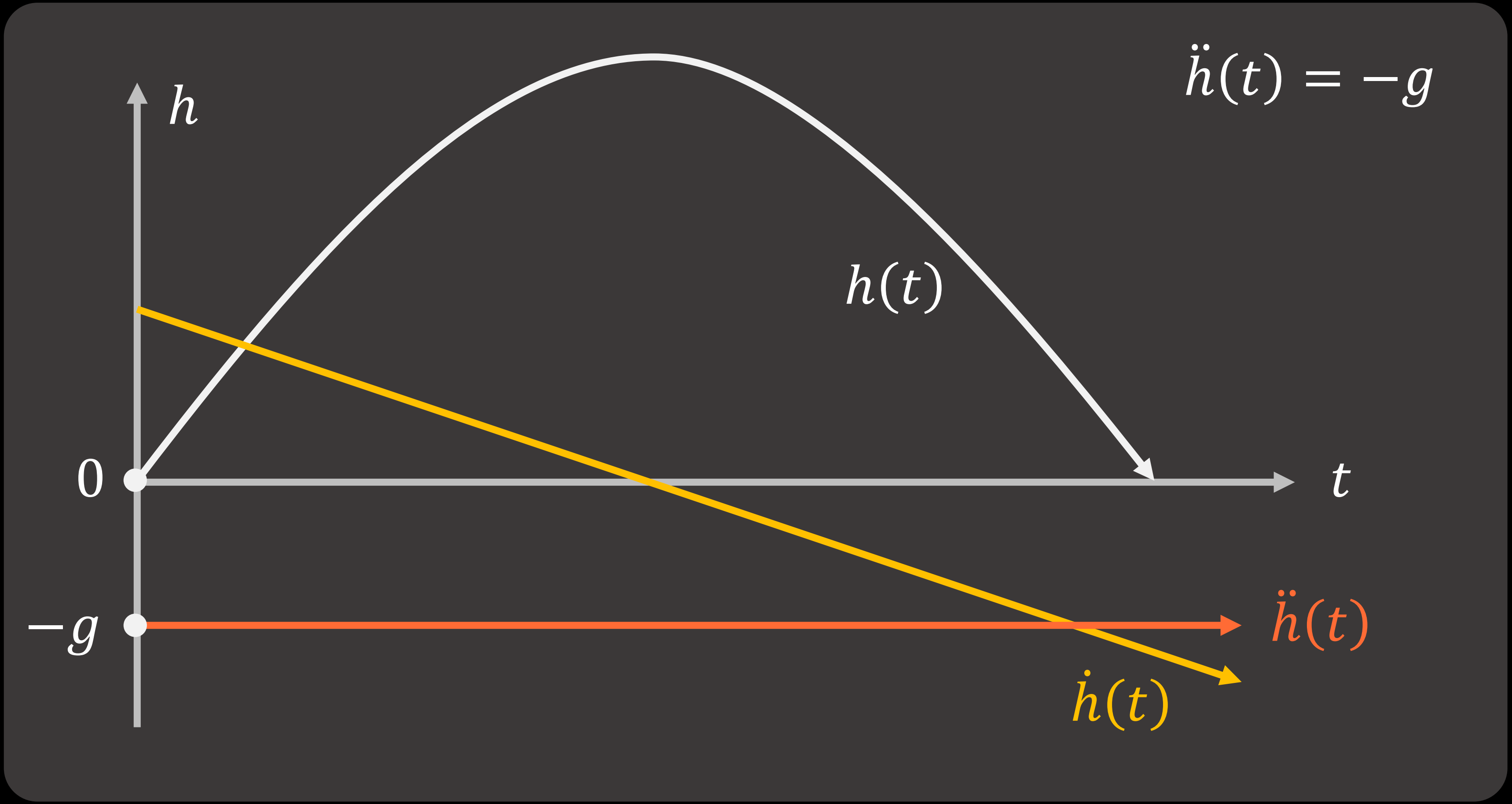
# The Soccer Equation

- $h(t)$ the height of the ball at time $t$

$$\ddot{h}(t) = -g$$

- The acceleration of the ball is $-g$ $\quad (g \approx 9{,}81\,\frac{m}{s^2})$
- Gravity pulls the velocity downwards

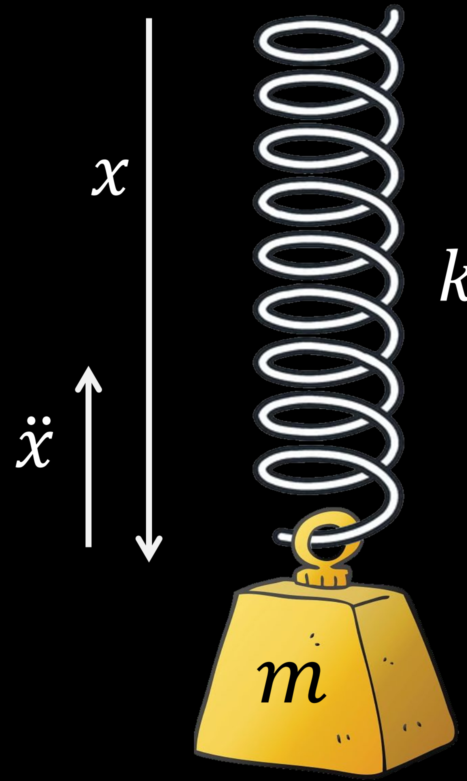
# Guessing the Solution



# The Spring Equation

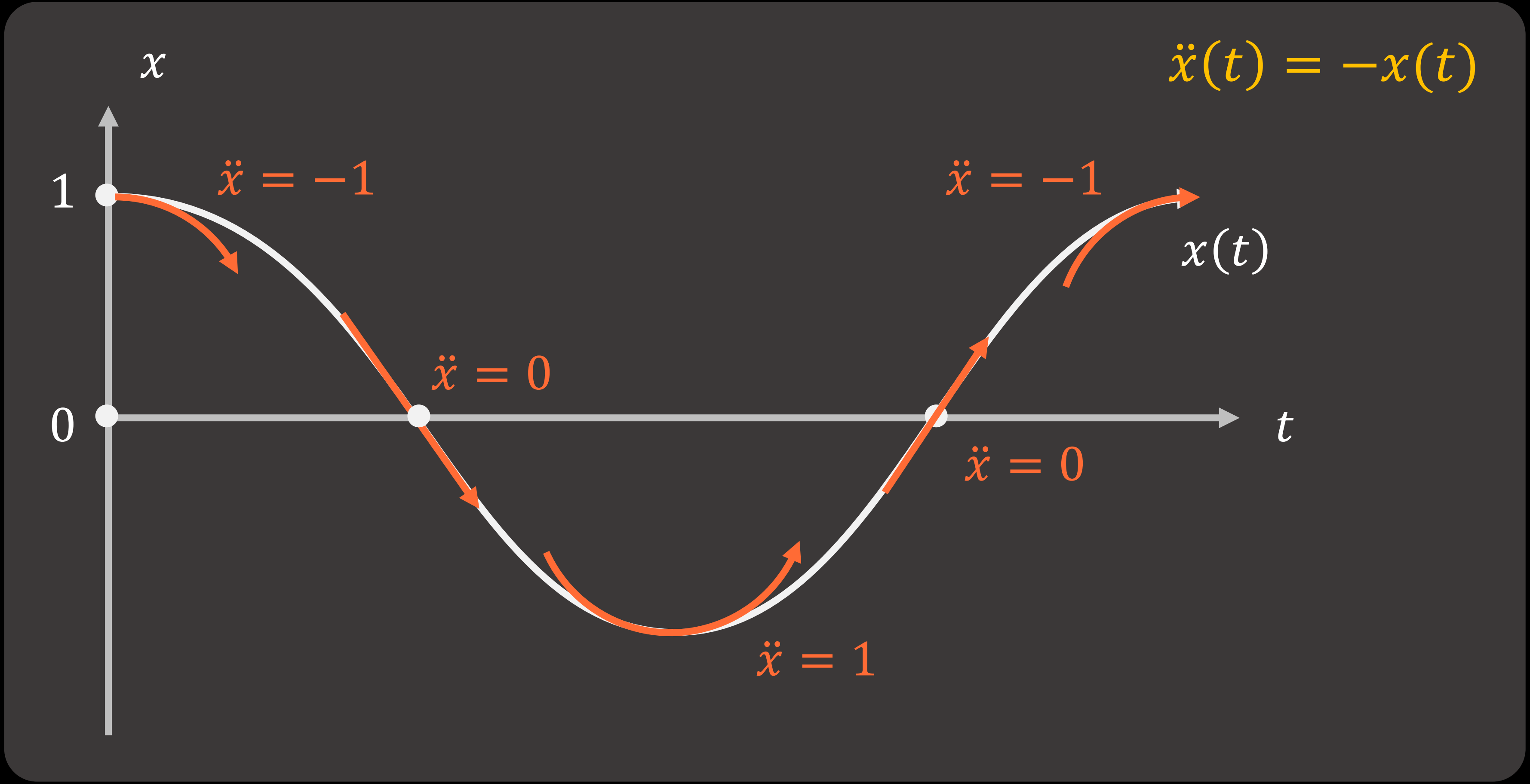
- $x(t)$ the elongation of the spring at time $t$

$$\ddot{x}(t) = -\frac{k}{m} \cdot x(t)$$



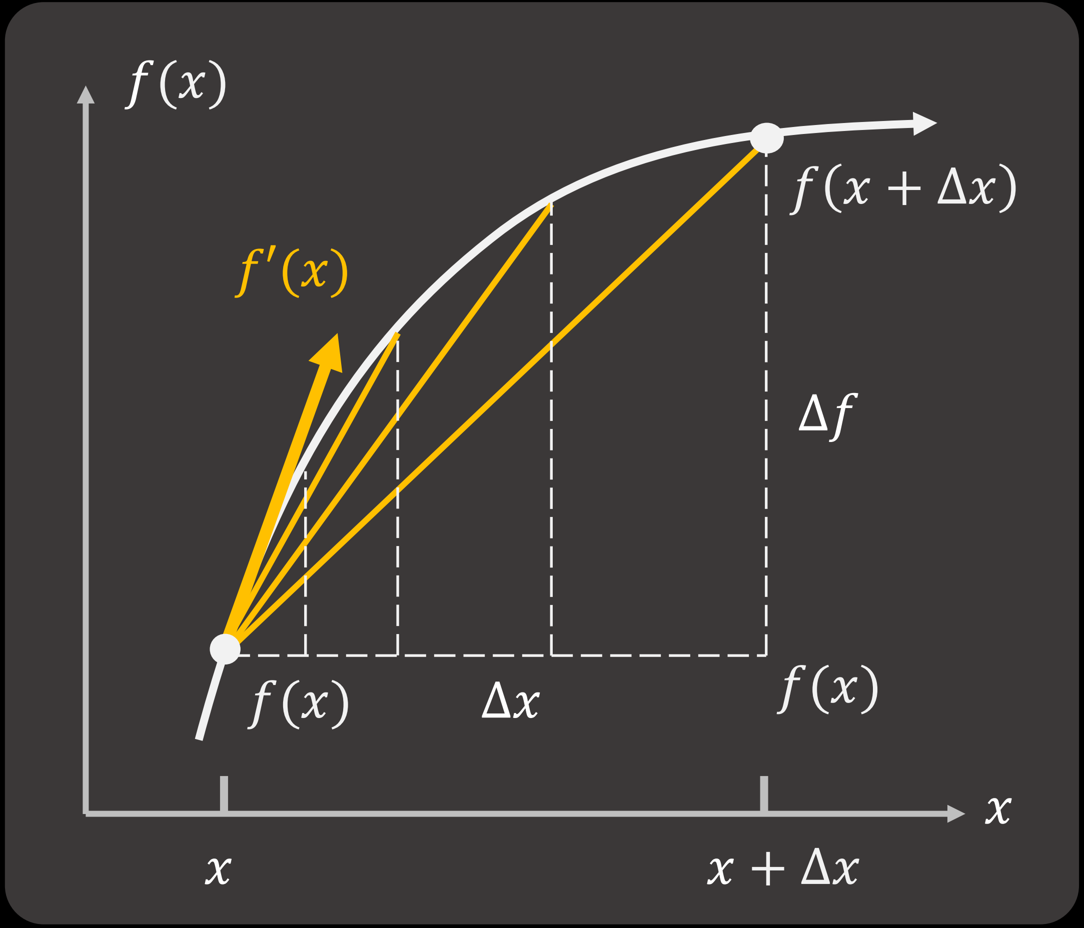
- Newton's law: $f = m \cdot \ddot{x}(t)$
- The Spring force: $f = -k \cdot x(t)$
- Zero gravity and zero rest length assumed

# Guessing the Solution



# How to solve differential equations mathematically

# Measuring a Changing Rate of Change



Estimate of the rate of change:

$$f'(x) \approx \frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Correct if $\Delta x = 0$, but then we have:

$$f'(x) = \frac{f(x) - f(x)}{0} = \frac{0}{0} = ?$$

Spawned a new branch of mathematics!

# Calculus

1642 - 1726

1646 - 1716

- Invented by Isaac Newton and Gottfried Leibniz
- Introduced an infinitely small but non-zero difference

$$\Delta x \longrightarrow dx$$

difference → differential

- Gives differential equations their name
- Strange quantity: How large is the value of $dx + dx$?
- Still used in mathematics today in a non-rigorous way as notation

# Using a Limit

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

- The instantaneous rate of change is the estimated rate of change as $\Delta x$ approaches but not reaches zero
- All you need to know to derive and understand all of calculus!

# Deriving the “Rules”

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f(x) = a \qquad f'(x) = \lim_{\Delta x \to 0} \frac{a - a}{\Delta x} = \boxed{0}$$

$$f(x) = x \qquad f'(x) = \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = \boxed{1}$$

# The Power Rule

- Tedious, not particularly difficult:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) \; - f(x)}{\Delta x}$$

$$f(x) = \boxed{x^n} \qquad f'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)(x + \Delta x)(x + \Delta x) \ldots (x + \Delta x) - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\Delta x + a_2 x^{n-2}\Delta x^2 + a_3 x^{n-3}\Delta x^3 + \cdots - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(nx^{n-1} + a_2 x^{n-2}\Delta x + a_3 x^{n-3}\Delta x^2 + \cdots\right) \qquad \boxed{= n \cdot x^{n-1}}$$

# Multiplication by Constant

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f(x) = a \cdot g(x)$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{a \cdot g(x + \Delta x) - a \cdot g(x)}{\Delta x}$$

$$= a \cdot \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} = a \cdot g'(x)$$

# The Sum Rule

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f(x) = f_1(x) + f_2(x)$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f_1(x+\Delta x) + f_2(x+\Delta x) - (f_1(x) + f_2(x))}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left( \frac{f_1(x+\Delta x) - f_1(x)}{\Delta x} + \frac{f_2(x+\Delta x) - f_2(x)}{\Delta x} \right)$$

$$= \lim_{\Delta x \to 0} \frac{f_1(x+\Delta x) - f_1(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{f_2(x+\Delta x) - f_2(x)}{\Delta x}$$

$$= f_1'(x) + f_2'(x)$$

# Polynomials

- Polynomials have the following form:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

- They are useful to approximate other functions (as we will see).

- Derived from previous rules:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

Back to the examples…

# The Soccer Equation

$$\ddot{h}(t) = -g$$

- If we choose:

$$h(t) = -\frac{1}{2}gt^2$$

$$\dot{h}(t) = -gt$$

$$\ddot{h}(x) = -g$$

- We can also choose :

$$h(t) = h_0 + v_0 t - \frac{1}{2}gt^2$$

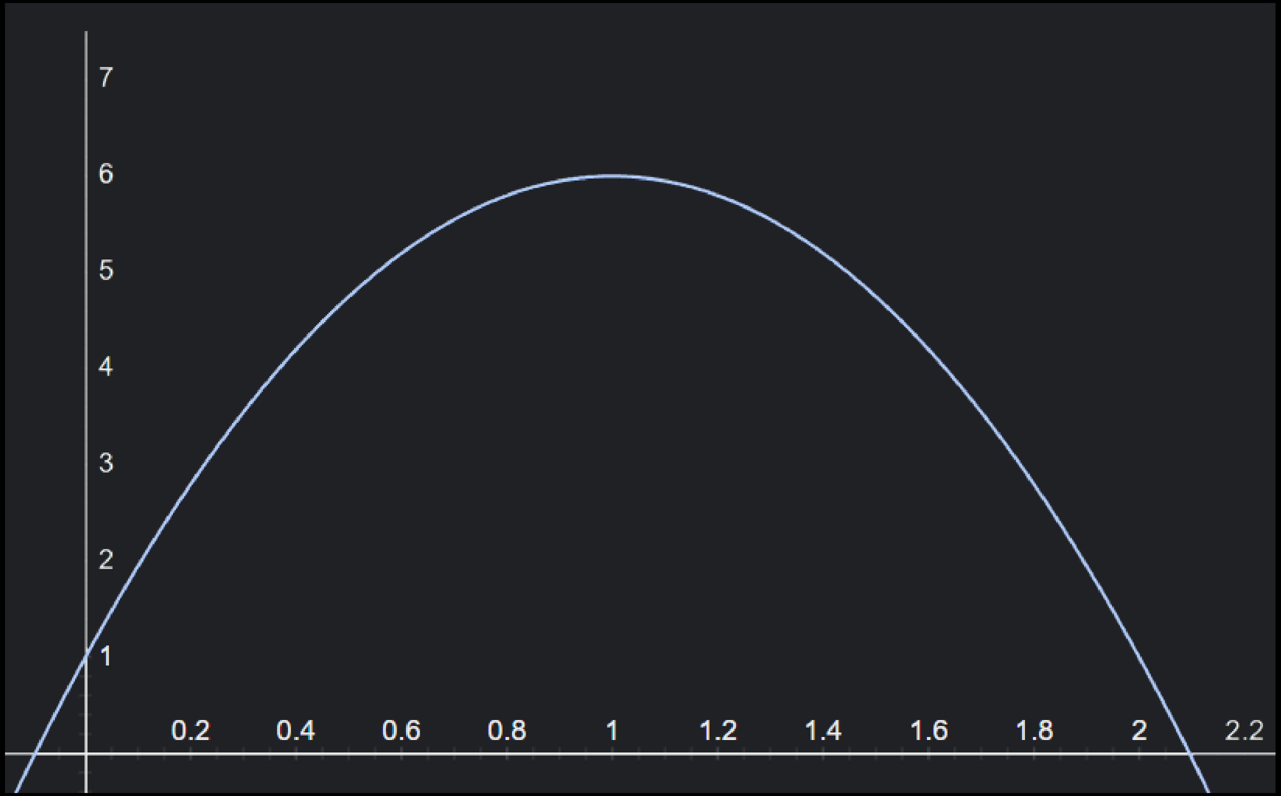
$$\dot{h}(t) = v_0 - gt$$

$$\ddot{h}(t) = -g$$

- Differential equations can have many solutions
- Here we can freely choose an initial height $h_0$ and an initial velocity $v_0$

# The Soccer Equation

$$h(t) = 1 + 10t - \frac{1}{2}10t^2$$



# Now comes the cool stuff!!

# The Rabbit Equation

$$\dot{N}(t) = k \cdot N(t)$$

- We need a function whose derivative is itself! (if $k = 1$)
- The motion, the velocity, the acceleration and the acceleration of the acceleration of the acceleration are all the same!
- $f(x) = x^n$ is not strong enough because $f'(x) = nx^{n-1}$
- Let's try: $\boxed{f(x) = e^x}$

# Exponents Recap

$$a^b \cdot a^c = \overbrace{a \cdot a \cdot a \cdot a}^{b} \cdot \overbrace{a \cdot a \cdot a \cdot a}^{c} = a^{b+c} \quad \rightarrow \quad \boxed{a^b \cdot a^c = a^{b+c}}$$

$$a^0 \cdot a = a^{0+1} = a^1 = a \quad \rightarrow \quad \boxed{a^0 = 1}$$

$$a^{-1} \cdot a = a^{-1+1} = a^0 = 1 \quad \rightarrow \quad \boxed{a^{-1} = 1/a}$$

$$a^{1/2} \cdot a^{1/2} = a^{1/2+1/2} = a^1 = a \quad \rightarrow \quad \boxed{a^{1/2} = \sqrt{a}}$$

# Finding $e$

$$(e^x)' = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^x\left(e^{\Delta x} - 1\right)}{\Delta x} = e^x \cdot \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x}$$

needs to be 1

- If we substitute $\Delta x = 1/n$ we need: $\lim_{\Delta n \to \infty} n\left(e^{1/n} - 1\right) = 1$
- For a finite $n$ can solve for an $e_n$ = $(1 + 1/n)^n$
- We are interested in $e = \lim_{n \to \infty} e_n$
- So $$e = \lim_{n \to \infty} (1 + 1/n)^n$$ .. a crazy formula!

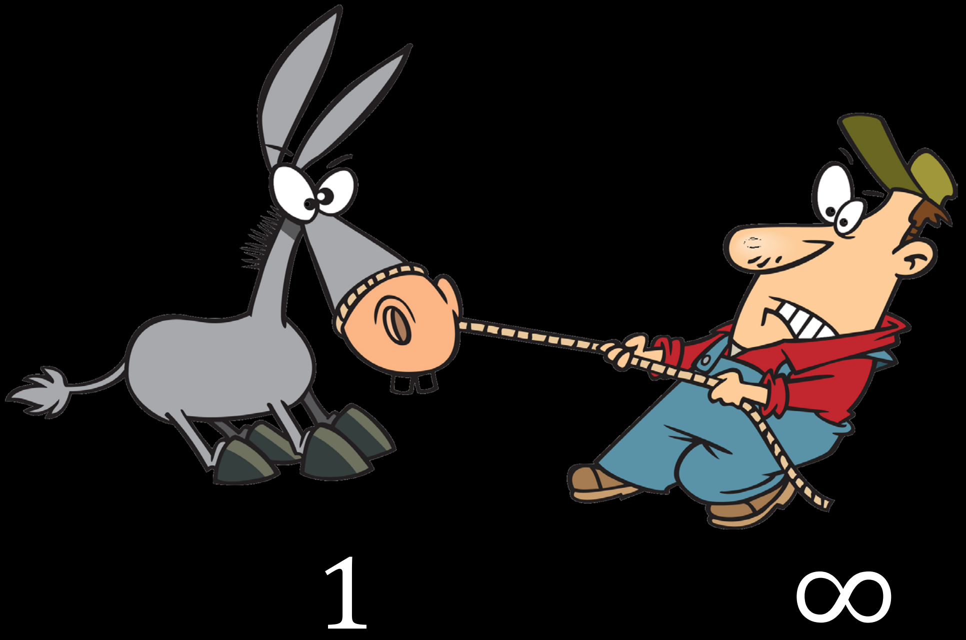
# The Crazy Formula

$$e = \lim_{n\to\infty} (1 + 1/n)^n$$

For $a > 1: \lim_{n\to\infty} a^n = \infty$

For $a = 1: \lim_{n\to\infty} a^n = 1$

Who wins?



Take your calculator and enter: `1.000001^1000000`

or this `1.0000000001^10000000000`

To this day, I cannot believe what I see!

# Approaching a “Random” Number

```
2.7182818284 5904523536 0287471352 6624977572 4709369995 9574966967
  6277240766 3035354759 4571382178 5251664274 2746639193 2003059921
  8174135966 2904357290 0334295260 5956307381 3232862794 3490763233
  8298807531 9525101901 1573834187 9307021540 8914993488 4167509244
  7614606680 8226480016 8477411853 7423454424 3710753907 7744992069
  5517027618 3860626133 1384583000 7520449338 2656029760 6737113200
  7093287091 2744374704 7230696977 2093101416 9283681902 5515108657
  4637721112 5238978442 5056953696 7707854499 6996794686 4454905987
  9316368892 3009879312 7736178215 4249992295 7635148220 82698 .....
```

- An infinite non-repeating sequence of digits
- The compromise between 1 and ∞

# An Alternative Approach

- Try a polynomial:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots$$

- For equality, we must have: $a_n = \frac{1}{n} a_{n-1}$

$$f(x) = 1 + 1 \cdot x + \frac{1}{2} x^2 + \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 4} x^4 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

- Only works for an infinitely long polynomial!

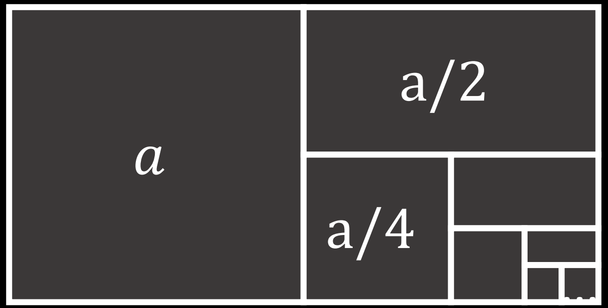
# Does it Explode?

$$f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2 \cdot 3}x^3 + \frac{1}{2 \cdot 3 \cdot 4}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

- For $x > 1$ the terms get bigger and bigger! – really?
- After $n > 2x$, the terms shrink by more than a factor of 2

$$\frac{x \cdot x \cdot x \cdot \cdots \cdot x}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1)} \cdot \frac{x}{n}$$

- Therefore, the sum for $n > 2x$ is limited:



# Are the Solutions Equal?

- They are equal at $x = 0$ because $f(0) = 1 = e^0$

- Let us assume they split at $x_s$, so: $f(x_s) = e^{x_s}$ but $f'(x_s) \neq (e^{x_s})'$
- However $f'(x_s) = f(x_s) = e^{x_s} = (e^{x_s})'$

- We have the beautiful result:

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n}$$

# Recomputing $e$

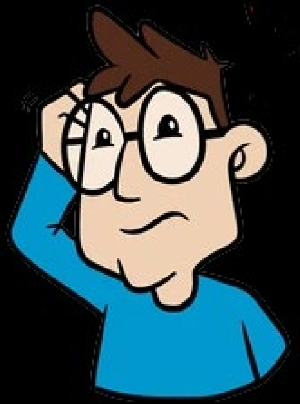
- Using the new equation:

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} 1^n = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} \approx 2.718$$

- Now the value of $e$ is somewhat more plausible (I think)

- And look at this crazy equation we derived we a bit of algebra:

$$\lim_{n\to\infty} (1 + 1/n)^n = \sum_{n=0}^{\infty} \frac{1}{n!}$$



# Generalization

$$f(ax) = 1 + ax + \frac{1}{2}a^2x^2 + \frac{1}{6}a^3x^3 + \frac{1}{24}a^4x^4 + \cdots$$

$$f'(ax) = a + a^2x + \frac{1}{2}a^3x^2 + \frac{1}{6}a^4x^3 + \cdots$$

$$= a \cdot f(ax)$$

$$\boxed{f(x) = e^{ax}}$$

$$\boxed{f'(x) = a \cdot e^{ax}}$$

# Ready for the last 3 equations

# The Rabbit Equation

$$\dot{N}(t) = k \cdot N(t)$$

$$N(t) = e^{kt}$$

because $\dot{N}(t) = k \cdot e^{kt} = k \cdot N(t)$

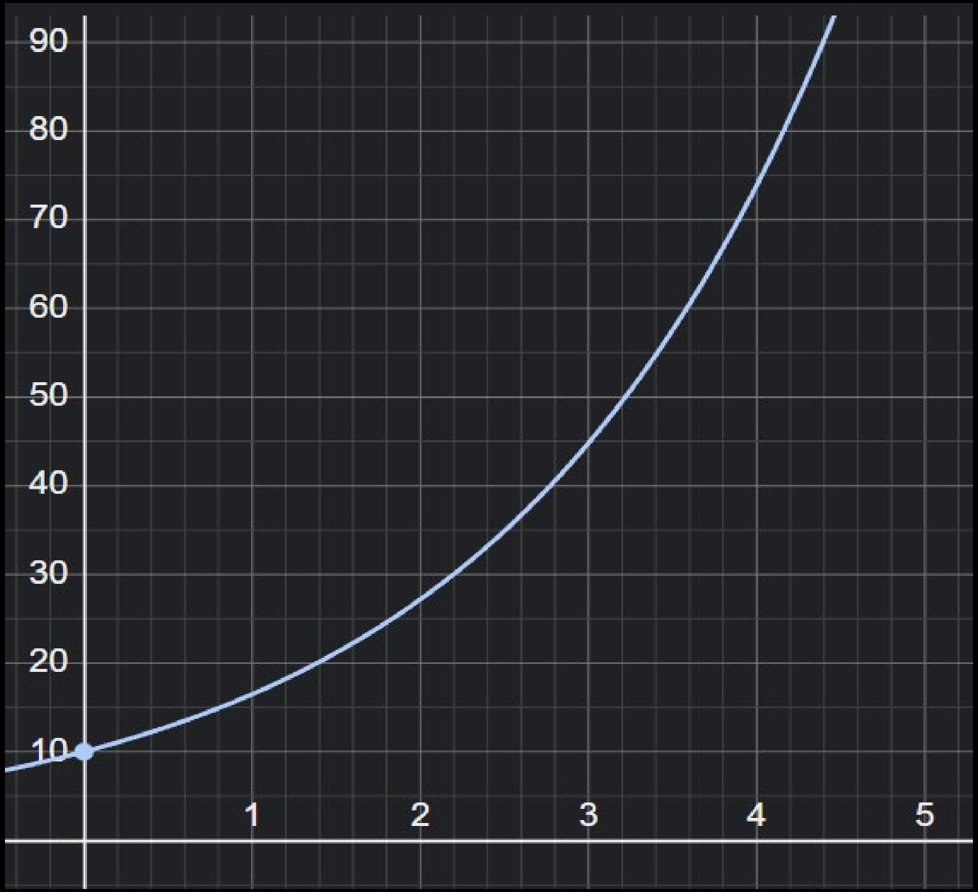
- More Solutions: we can choose the value of $N(t)$ at time $t_0$ :

$$N(t) = N_0 \cdot e^{k(t-t_0)}$$

$$= \left(N_0 \cdot e^{-kt_0}\right) \cdot e^{kt}$$

$$\dot{N}(t) = k\left(N_0 \cdot e^{-kt_0}\right) \cdot e^{kt} = k \cdot N(t)$$

# The Rabbit Equation



$$N(t) = 10e^{t/2}$$

# The Coffee Equation

$$\dot{T}(t) = -k \cdot T(t)$$

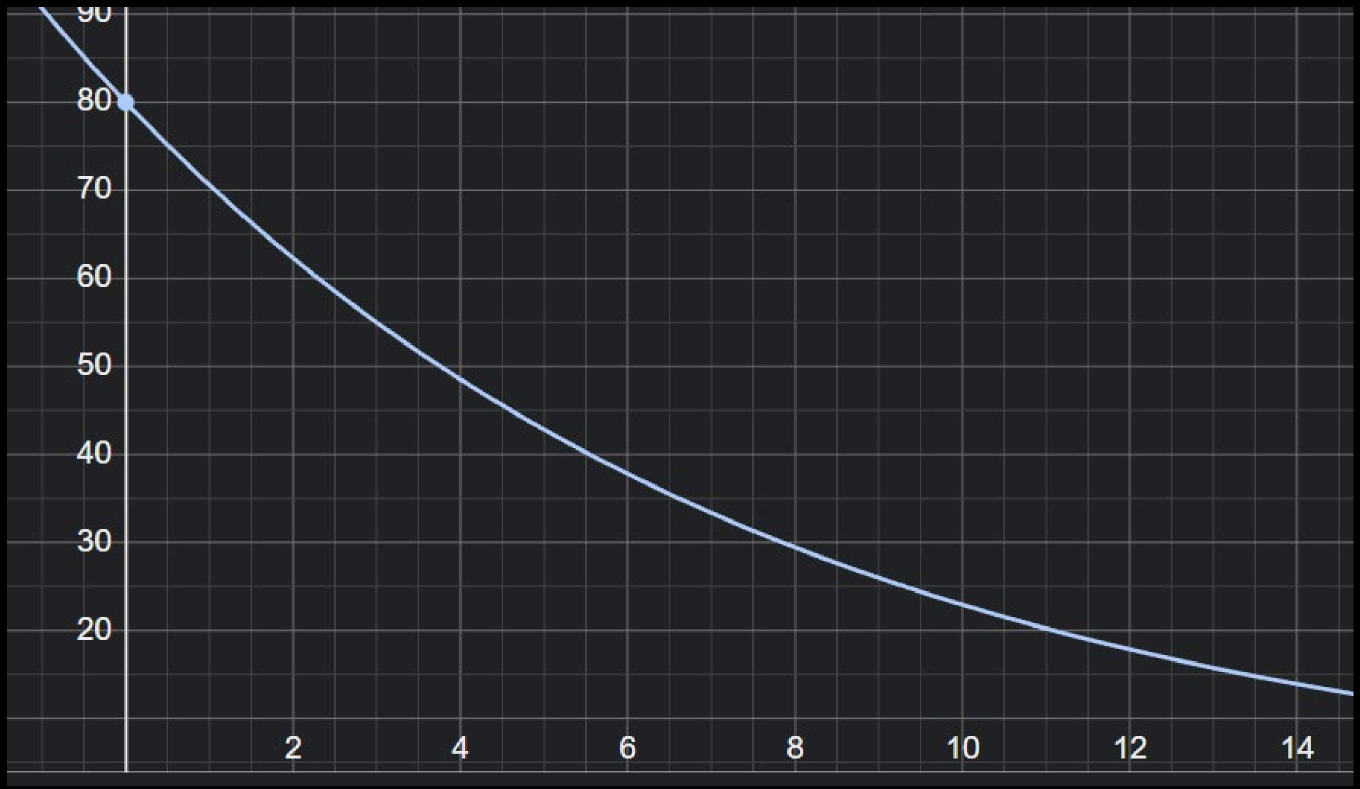
$$\boxed{T(t) = e^{-kt}}$$ because $\dot{T}(t) = -k \cdot e^{-kt} = -k \cdot T(t)$

- More Solutions: we can choose the value of $T(t)$ at time $t_0$ :

$$\boxed{T(t) = T_0 \cdot e^{-k(t-t_0)}}$$

# The Coffee Equation

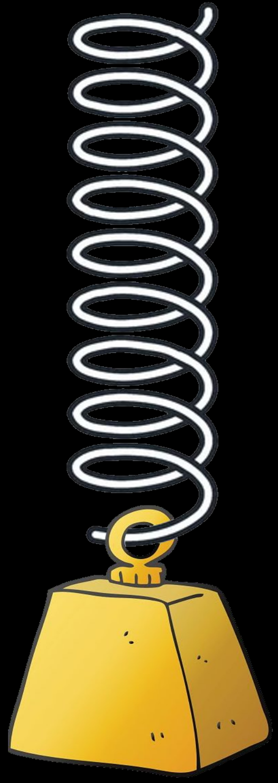
$$T(t) = 80e^{-t/8}$$



# The Spring Equation

$\ddot{x}(t) = -x(t)$

($k = 1$ for simplicity)



- If we try $x(t) = e^{at}$ as before we get: $\dot{x}(\mathrm{t}) = ae^{at}$ and $\ddot{x}(\mathrm{t}) = a^2e^{at}$
- Therefore, we must have:

$$a^2e^{at} = -e^{at}$$
$$a^2 = -1$$
$$a = \sqrt{-1}$$

- Oops, the square root of a negative number does not exist!

# The Imaginary Number

- Invent a new number $i = \sqrt{-1}$ (the imaginary number)
- Our solution: $x(t) = e^{it}$ $\quad x(t) = e^{i\sqrt{k}t}$ if $k \neq 1$
- Is this solution useful? What does it mean?

# Juggling with $i$

- From the definition:

$$i^0 = 1 \quad i^1 = i \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1 \quad i^5 = i \quad i^6 = -1 \quad \dots$$

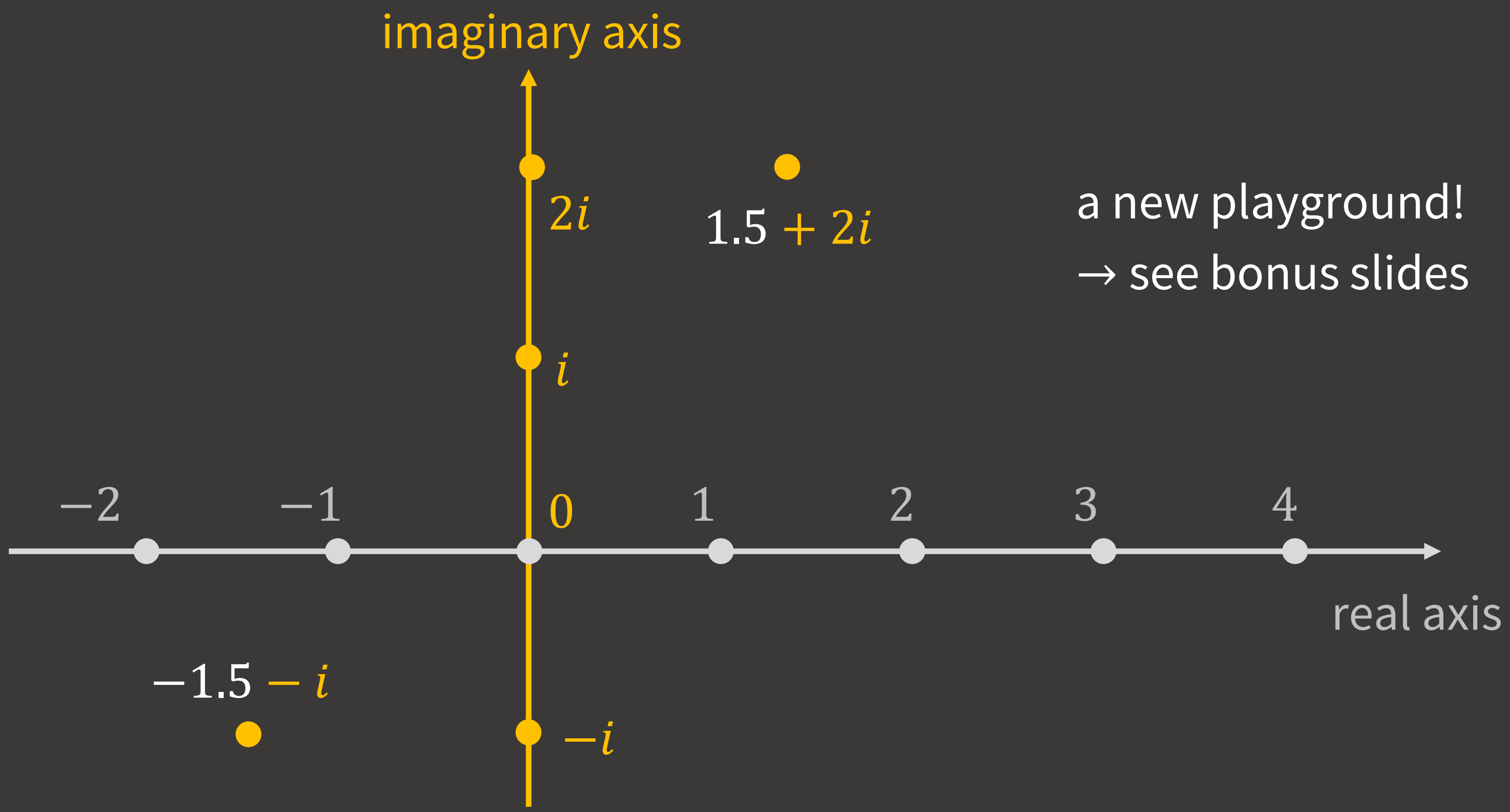
- The expression $a + b \cdot i$ cannot be further reduced

- Therefore, we define a complex number to be $z = \underset{\text{real part}}{a} + \underset{\text{imaginary part}}{b} \cdot i$

- Use regular algebra, e. g. for multiplication:

$$(a_1 + b_1 i) \cdot (a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) \cdot i$$

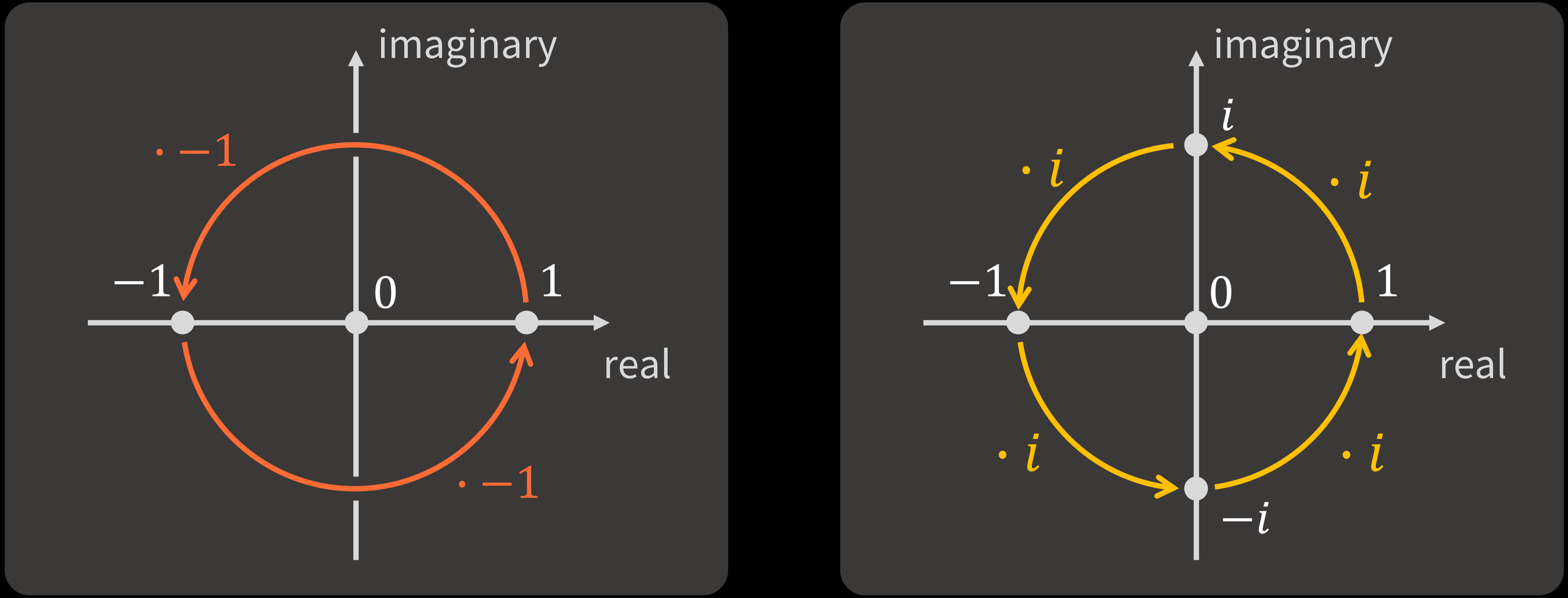
# The Complex Plane



a new playground!

→ see bonus slides

# Being the Square Root of -1



- Multiplying with $i$ results in a half step from 1 to -1 but not going through zero!
- For this we needed the new dimension

# Plugging it in

$$\boxed{e^{ix}} = 1 + ix + \frac{1}{2!}i^2x^2 + \frac{1}{3!}i^3x^3 + \frac{1}{4!}i^4x^4 + \frac{1}{5!}i^5x^5 + \frac{1}{6!}i^6x^6 + \cdots$$

$$= 1 + ix + \frac{1}{2!}(-1)x^2 + \frac{1}{3!}(-i)x^3 + \frac{1}{4!}1x^4 + \frac{1}{5!}ix^5 + \frac{1}{6!}(-1)x^6 + \cdots$$

$$= \boxed{\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right)} + i\,\boxed{\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right)}$$

# Checking...

$$f(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots$$

$$\dot{f}(x) = -x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \frac{1}{7!}x^7 - \cdots$$

$$\ddot{f}(x) = -1 + \frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \frac{1}{6!}x^6 - \cdots = -f(t)$$

$$f(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

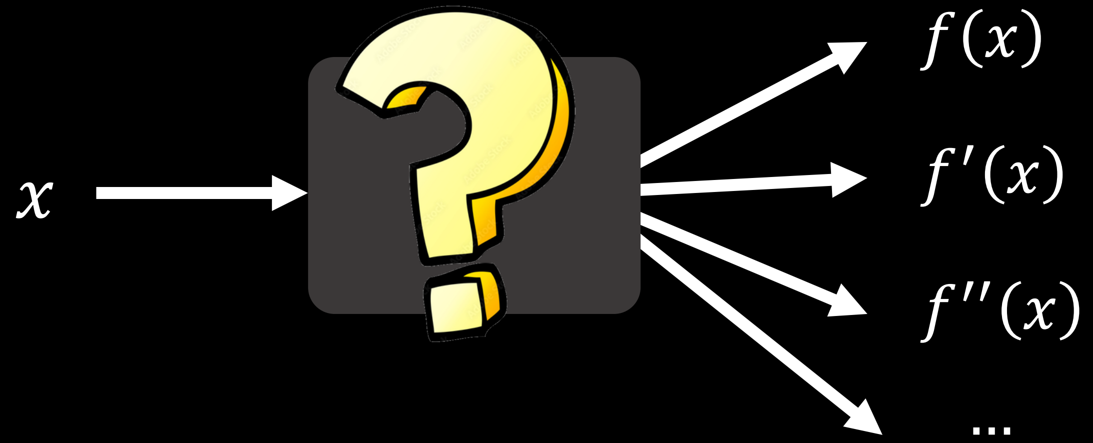
$$\dot{f}(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

$$\ddot{f}(x) = -x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \cdots = -f(x)$$

- Both work!
- Do they have a closed form like $e^{ix}$?
- Do they have a deeper meaning, interpretation?

# The Polynomial Quiz

- Given a polynomial in a black box with unknown coefficients $a_i$
- We are allowed to evaluate it and all its derivatives



$x$

$f(x)$

$f'(x)$

$f''(x)$

…

- Can we recover the coefficients?

# Solution

$$
\begin{array}{rcccccccccccc}
f(x) = & a_0 & + & a_1 x & + & a_2 x^2 & + & a_3 x^3 & + & a_4 x^4 & + & a_5 x^5 & + \cdots \\
f'(x) = & & & a_1 & + & 2a_2 x & + & 3a_3 x^2 & + & 4a_4 x^3 & + & 5a_4 x^4 & + \cdots \\
f''(x) = & & & & & 2a_2 & + & 6a_3 x & + & 12a_4 x^2 & + & 20a_4 x^3 & + \cdots \\
f'''(x) = & & & & & & & 6a_3 & + & 24a_4 x & + & 60a_4 x^2 & + \cdots \\
f''''(x) = & & & & & & & & & 24a_4 & + & 120a_4 x & + \cdots
\end{array}
$$

Evaluate at zero, solve for the only remaining coefficient:

$$a_n = \frac{f^{(n)}(0)}{n!}$$

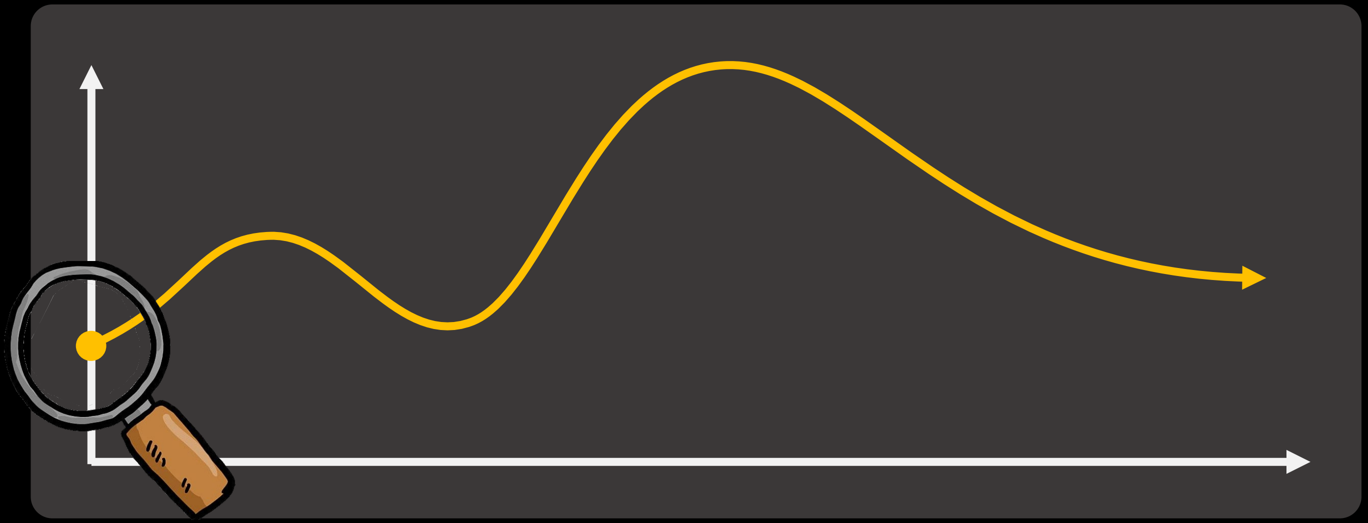
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

*Taylor expansion* at zero

*Maclaurin series*

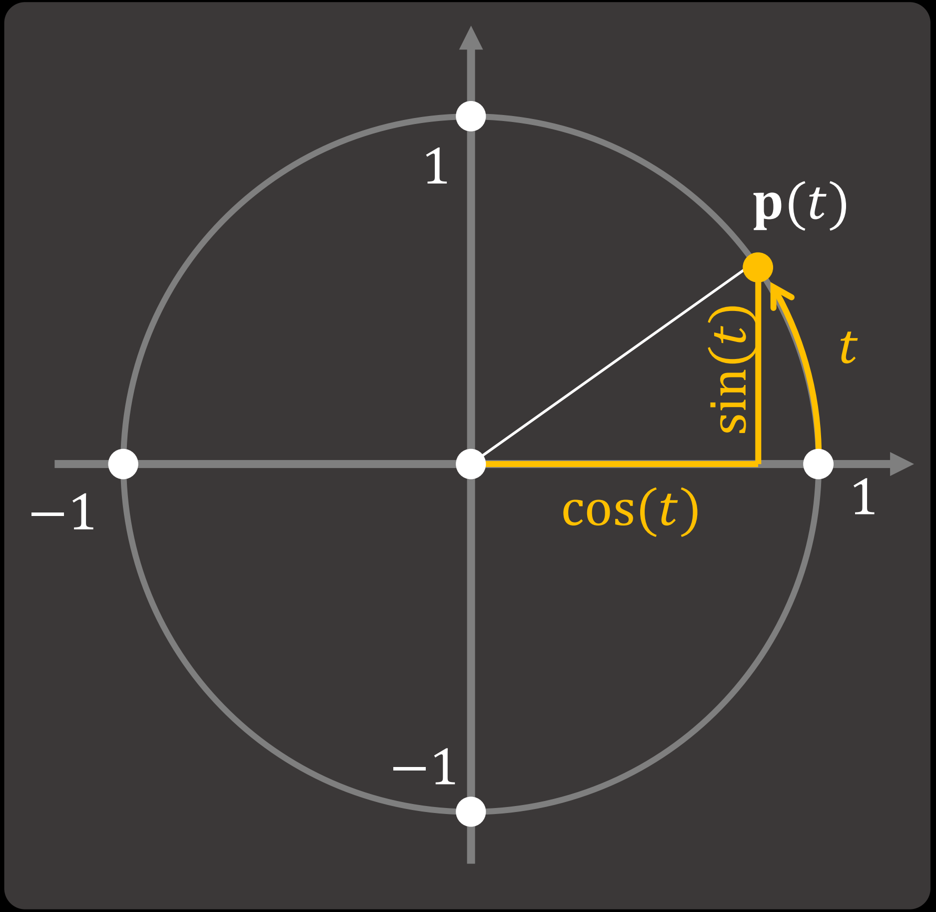
# The Big Bang Theory

- *If a function is analytic everywhere with no singularities,*
  *then the Maclaurin series converges to the function for all values of $x$.*



- It is enough to inspect the function at zero (big bang) to recover its entire shape!

# Point on a Circle

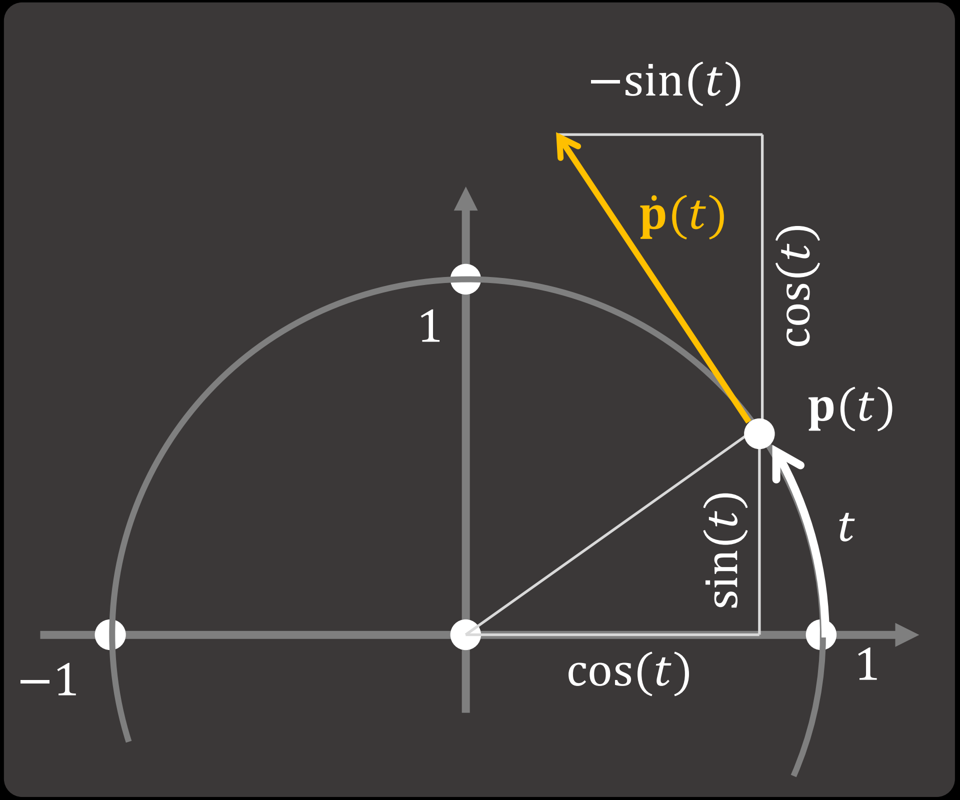


- Point moving on a unit circle with unit velocity
- Giving the coordinate functions names:

$$\mathbf{p}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

- The definition of sine and cosine

# The Derivative



- The time derivative of the path is the velocity.
- Tangent to the circle, length 1

$$\dot{\mathbf{p}}(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$$

- Therefore

$$\cos'(t) = -\sin(t)$$
$$\sin'(t) = \cos(t)$$

# Euler's Magic Formula

| | |
|---|---|
| $\cos(0) = \cos(0) = 1$ | $\sin(0) = \sin(0) = 0$ |
| $\cos'(0) = -\sin(0) = 0$ | $\sin'(0) = \cos(0) = 1$ |
| $\cos''(0) = -\cos(0) = -1$ | $\sin''(0) = -\sin(0) = 0$ |
| $\cos'''(0) = \sin(0) = 0$ | $\sin'''(0) = -\cos(0) = -1$ |
| $\cos''''(0) = \cos(0) = 1$ | $\sin''''(0) = \sin(0) = 0$ |
| … | … |

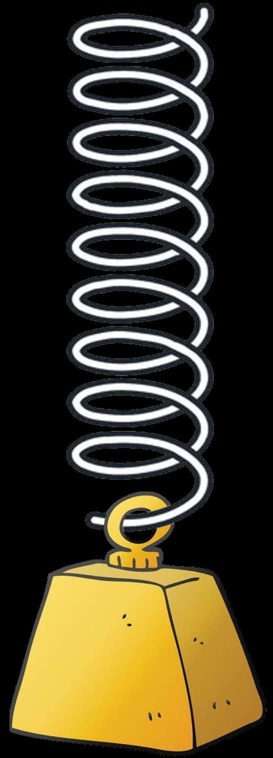
- Our earlier result:

$$e^{ix} = \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right) + i\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right)$$

- These are the Maclaurin series of $\cos(x)$ and $\sin(x)$ ! Therefore, we get:

$$e^{ix} = \cos(x) + i\sin(x)$$

# The Spring Equation



$$\ddot{x}(t) = -x(t)$$

$$\ddot{x}(t) = -\frac{k}{m} \cdot x(t)$$

$$x(t) = e^{it}$$

$$x(t) = e^{i\sqrt{k}t}$$

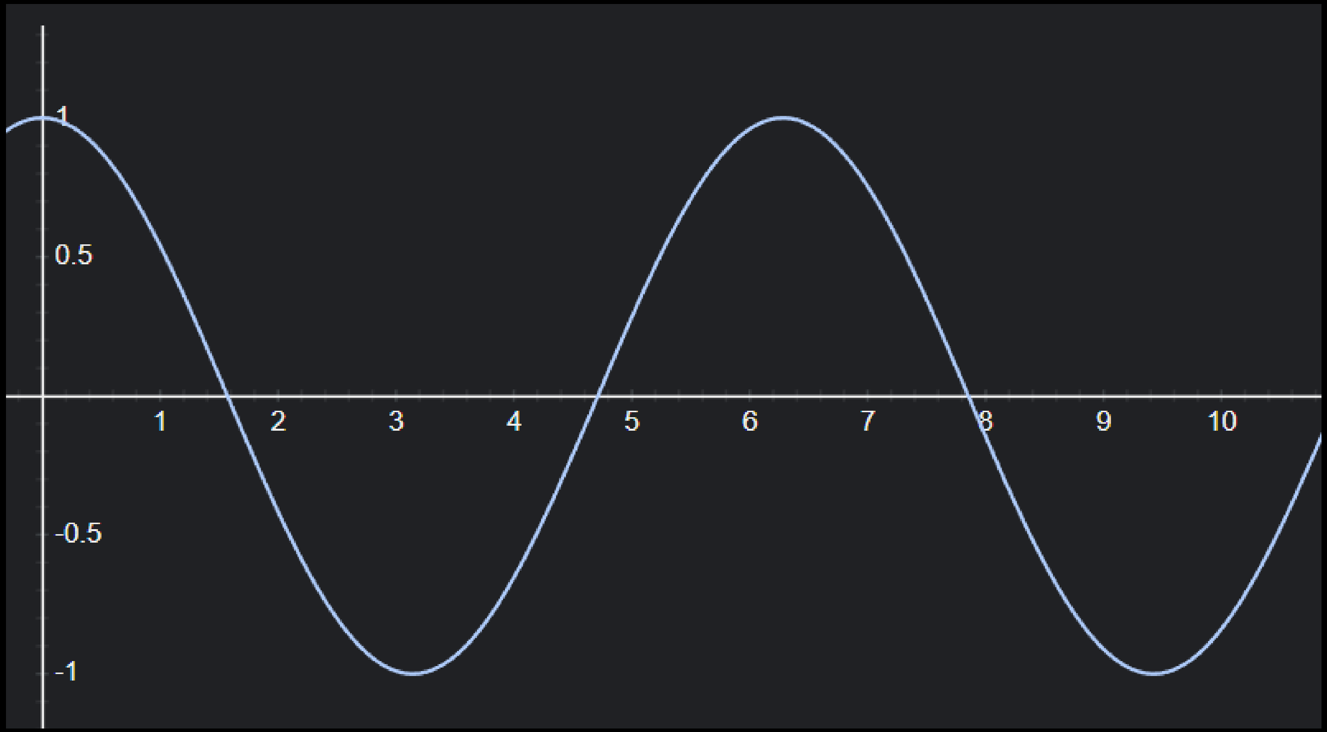
$$x(t) = \cos(\sqrt{k/m} \cdot t)$$

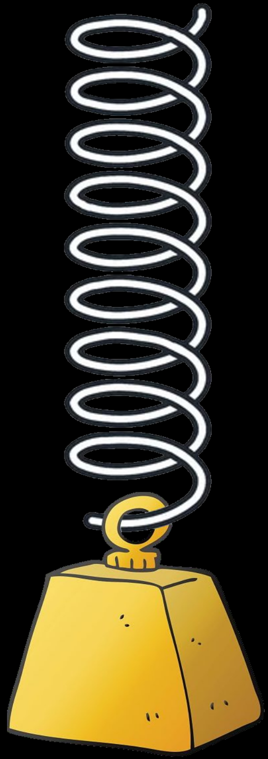
- The general solution:

$$x(t) = \mathrm{a} \cdot \sin(\sqrt{k/m} \cdot t) + \mathrm{b} \cdot \cos(\sqrt{k/m} \cdot t)$$

# The Spring Equation

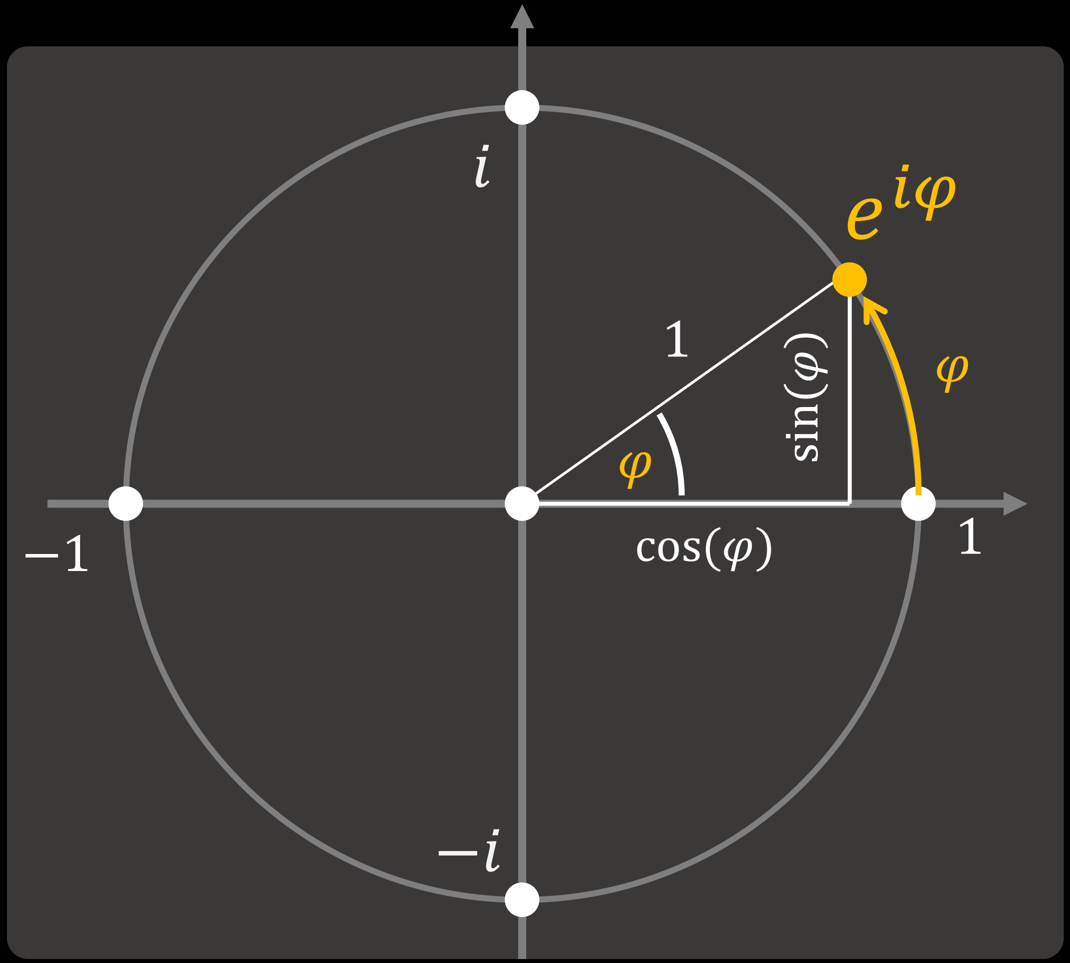
$$x(t) = \cos(t)$$






BONUS

# Euler in the Complex Plane

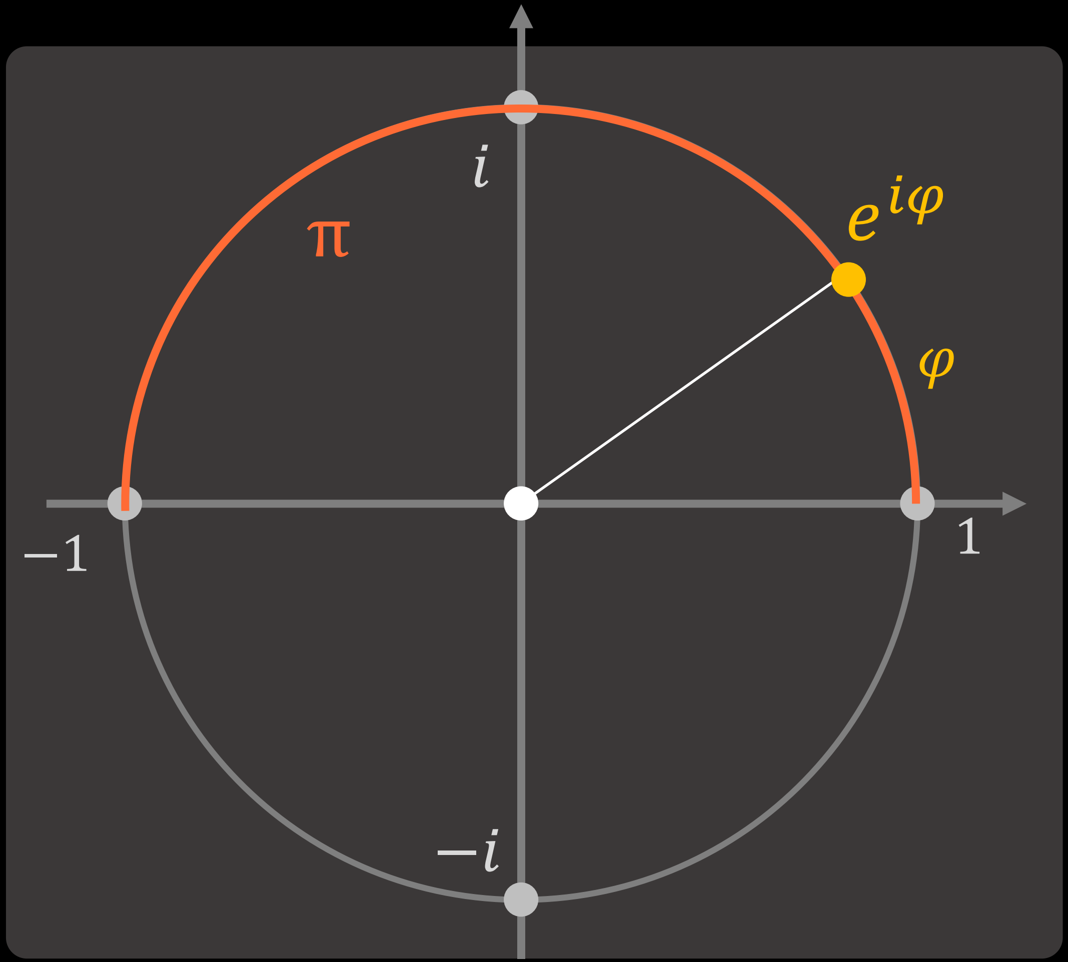


- Euler's equation states:

$$e^{i\varphi} = \cos(\varphi) + i \sin(\varphi)$$

- Therefore, the complex number $e^{i\varphi}$
  - Lies on the unit circle
  - The variable $\varphi$ is the distance from the number 1 along the unit circle
  - The definition of an angle in radians

# Introducing $\pi$



- The constant $\pi$ is defined to be:

  Half the circumference of the unit circle

- Now we get:

$$e^0 = 1$$
$$e^{i\pi/2} = i$$
$$e^{i\pi} = -1$$

★ famous, not special

$$e^{i3\pi/2} = -i$$
$$e^{i2\pi} = 1$$

# $\pi$

```
3.1415926535 8979323846 2643383279 5028841971 6939937510 5820974944 5923078164
  0628620899 8628034825 3421170679 8214808651 3282306647 0938446095 5058223172
  5359408128 4811174502 8410270193 8521105559 6446229489 5493038196 4428810975
  6659334461 2847564823 3786783165 2712019091 4564856692 3460348610 4543266482
  1339360726 0249141273 7245870066 0631558817 4881520920 9628292540 9171536436
  7892590360 0113305305 4882046652 1384146951 9415116094 3305727036 5759591953
  0921861173 8193261179 3105118548 0744623799 6274956735 1885752724 8912279381
  8301194912 9833673362 4406566430 8602139494 6395224737 1907021798 6094370277
  0539217176 2931767523 8467481846 7669405132 0005681271 4526356082 7785771342
  ...
```

- An infinite non-repeating sequence of digits
- We have the crazy result: $e^{i\pi} = -1$

⬇

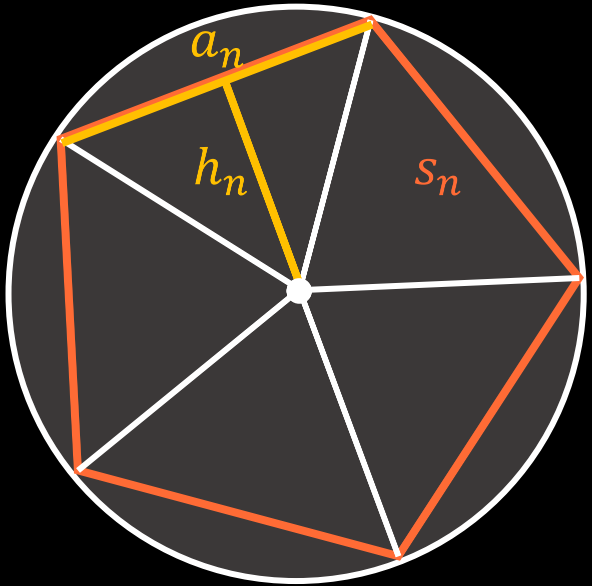
$$2.71828\ 18284\ 59045\ 23536..^{\sqrt{-1}\cdot 3.14159\ 26535\ 89793\ 23846..} = -1$$

# Computing the Digits of $\pi$

| Year | Who | How | Digits |
|---|---|---|---|
| 1400 | Madhava of Sangamagrama | By hand | 10 |
| 1706 | John Machin | By hand | 100 |
| 1949 | Levi B. Smith and John Wrench | Desk Calculator | 1000 |
| 1958 | Francois Genuys | IBM 704 | 10,000 |
| 1961 | Daniel Shanks and John Wrench | IBM 7090 | 100,000 |
| 1973 | Jean Guilloud and Martine Bouyer | CDC 7600 | 1,000,000 |
| 1983 | Yasunori Ushiro and Yasumasa Kanada | HITAC S-810/20 | 10,000,000 |
| 1987 | Yasumasa Kanada at al. | NEC SX-2 | 100,000,000 |
| 1989 | Gregory Chudnovsky & David Chudnovsky | IBM 3090 | 1,000,000,000 |
| 2019 | Emma Haruka Iwao | n1-megamem-96 | 31,415,926,535,897 |
| … | | | |

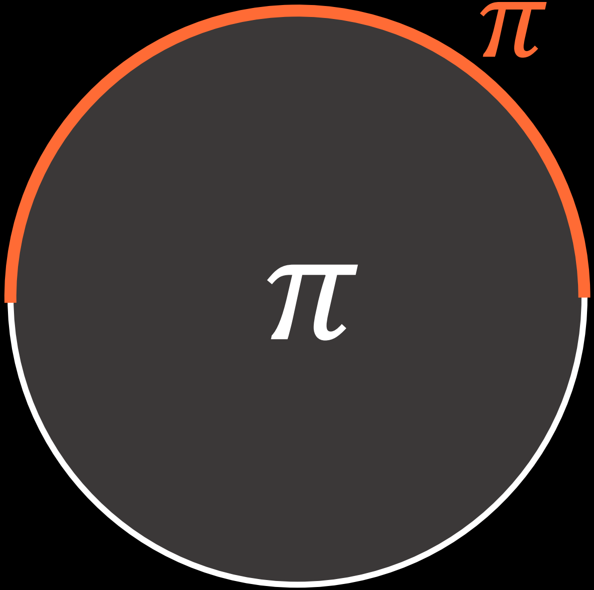
# More $\pi$

- Area of an $n$-gon:



$$A_n = n(\frac{1}{2}a_n h_n) \;= \frac{1}{2}s_n h_n$$

- Area of the unit disk:



$$A_{disc} = \lim_{n\to\infty}\frac{1}{2}s_n h_n \;= \frac{1}{2}2\pi \cdot 1 = \pi$$

# More $i$

- What is the value of $i^i$ ?
- Basic algebra:

$$a^b \cdot a^c = a^{b+c}$$

$$\left(a^b\right)^c = \underbrace{a^b \cdot a^b \cdot a^b \cdot a^b \cdot a^b}_{c} = a^{c \cdot b}$$

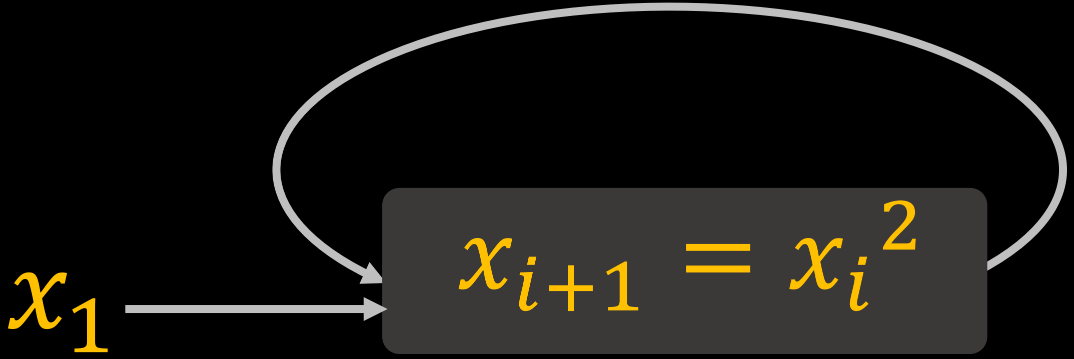
- Therefore:

$$i^i = \left(e^{\frac{\pi}{2}i}\right)^i = e^{\frac{\pi}{2}i \cdot i} = e^{-\frac{\pi}{2}} = \boxed{0.20787957635}$$

- Some “random” real number! – how crazy is math?

# The Square Game

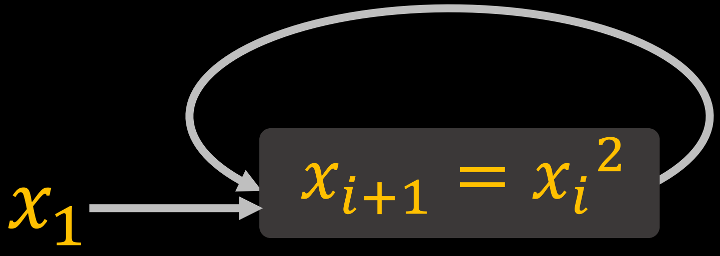
- Start with any number
- Keep squaring



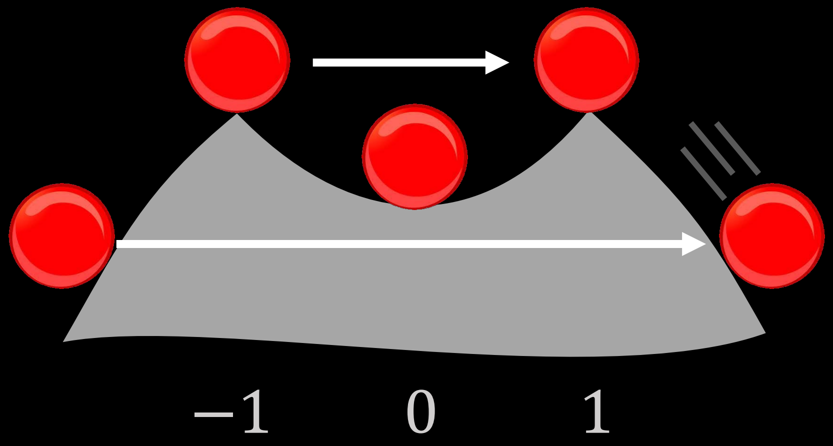
$x_1$

$x_{i+1} = {x_i}^2$

# Playing the Game



$x_1 \rightarrow \boxed{x_{i+1} = {x_i}^2}$

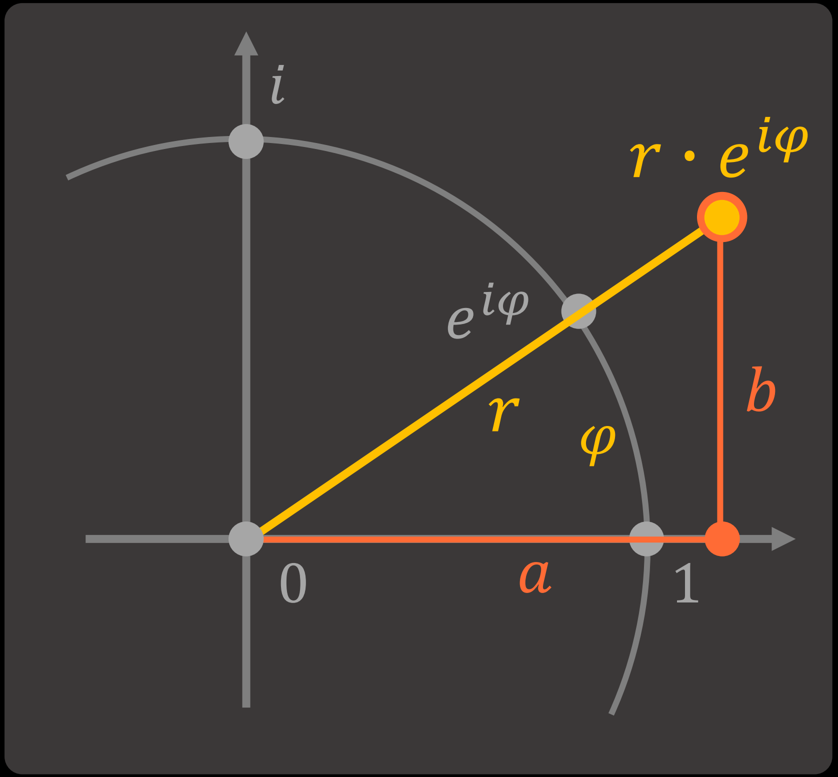


- Enter a number into your calculator
- Keep pressing $\boxed{x^2}$

| | |
|---|---|
| $-1 < x < 1$ | rolling into the sink at zero |
| $x > 1, < -1$ | rolling into the abyss |
| $x = -1, 1$ | staying on the edge |

# Where is the edge in the complex plane?

# Polar Coordinates



- For all $z = a + b \cdot i$ we can find $r, \varphi$ such that:

$$\boxed{a + b \cdot i} = \boxed{r \cdot e^{i\varphi}}$$

# Squaring Complex Numbers

- Polar representation

$$z^2 = r \cdot e^{i\varphi}$$

$$z^2 = r \cdot e^{i\varphi} \cdot r \cdot e^{i\varphi} = r^2 \cdot e^{i2\varphi}$$

- Squares the radius, doubles the angle

- Regular representation

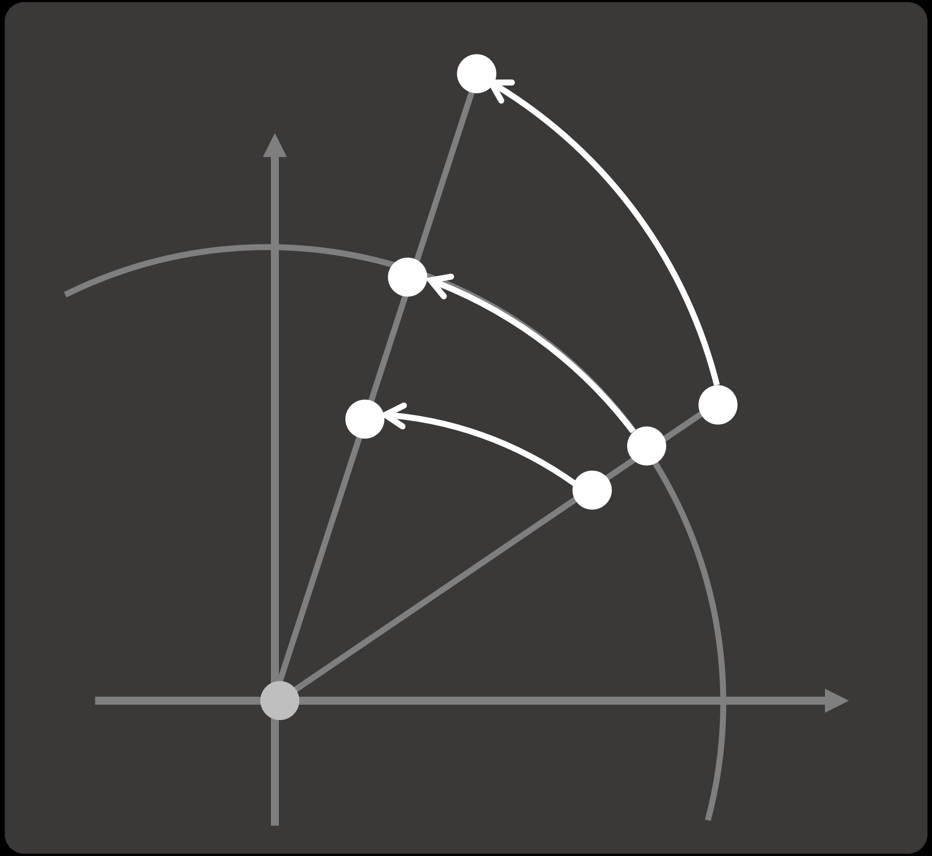
$$z = a + bi$$

$$z^2 = (a + bi) \cdot (a + bi)$$

$$a^2 + 2abi + b^2 i^2$$

$$a^2 + 2abi - b^2$$

$$(a^2 - b^2) + (2ab)i$$



# The Edge

- Squaring a complex number:

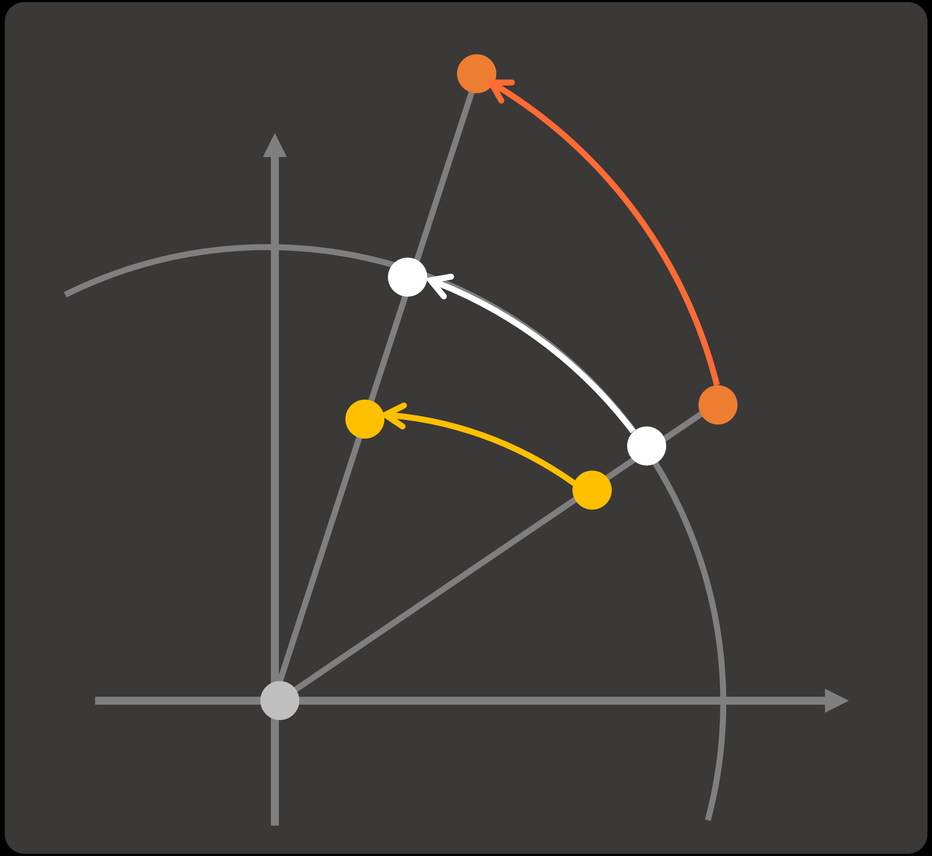
$$z^2 = r \cdot e^{i\varphi} \cdot r \cdot e^{i\varphi} = r^2 \cdot e^{i2\varphi}$$

- Squares the radius, doubles the angle

$r < 1$: points swirl towards the sink at 0

$r = 1$: points rotate on the unit circle

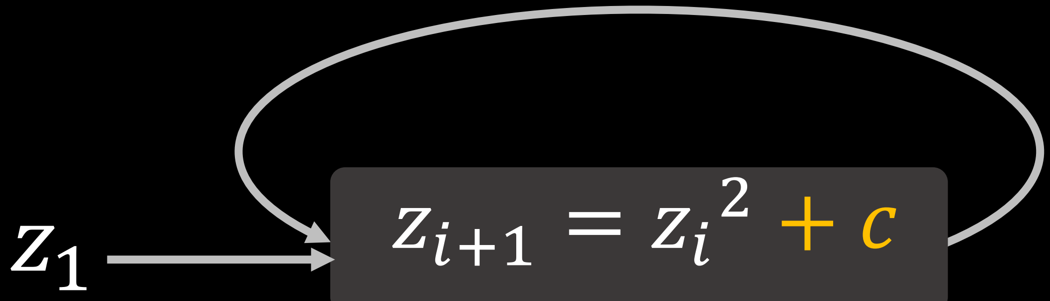
$r > 1$: points swirl towards infinity
the game diverges



- The edge is the unit circle!

# Introducing Wind

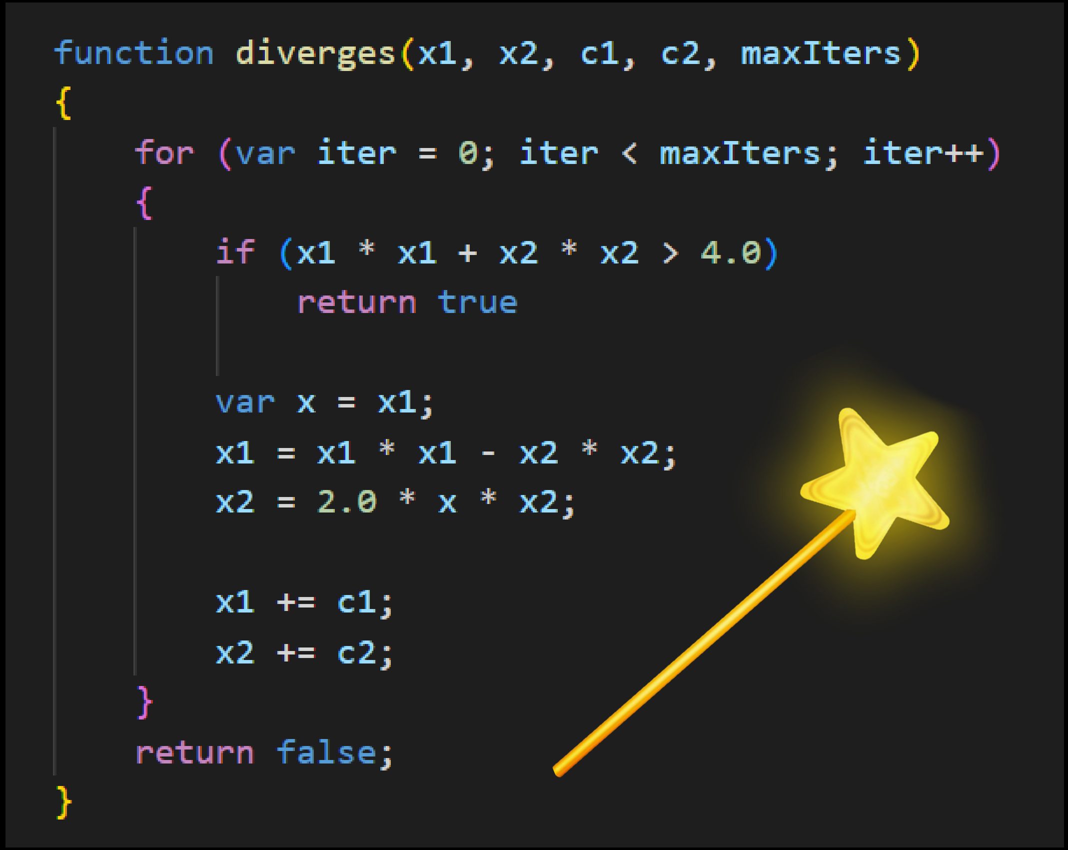
- What if the ball is pushed by a constantly blowing wind
- After each iteration we add a constant (complex) number $c$ :



$$z_1 \rightarrow \boxed{z_{i+1} = {z_i}^2 + c}$$

- Where is the edge now?
- What happens to the circle?

# Let’s write a Program



```
function diverges(x1, x2, c1, c2, maxIters)
{
    for (var iter = 0; iter < maxIters; iter++)
    {
        if (x1 * x1 + x2 * x2 > 4.0)
            return true

        var x = x1;
        x1 = x1 * x1 - x2 * x2;
        x2 = 2.0 * x * x2;

        x1 += c1;
        x2 += c2;
    }
    return false;
}
```

play the game for $n$ iterations

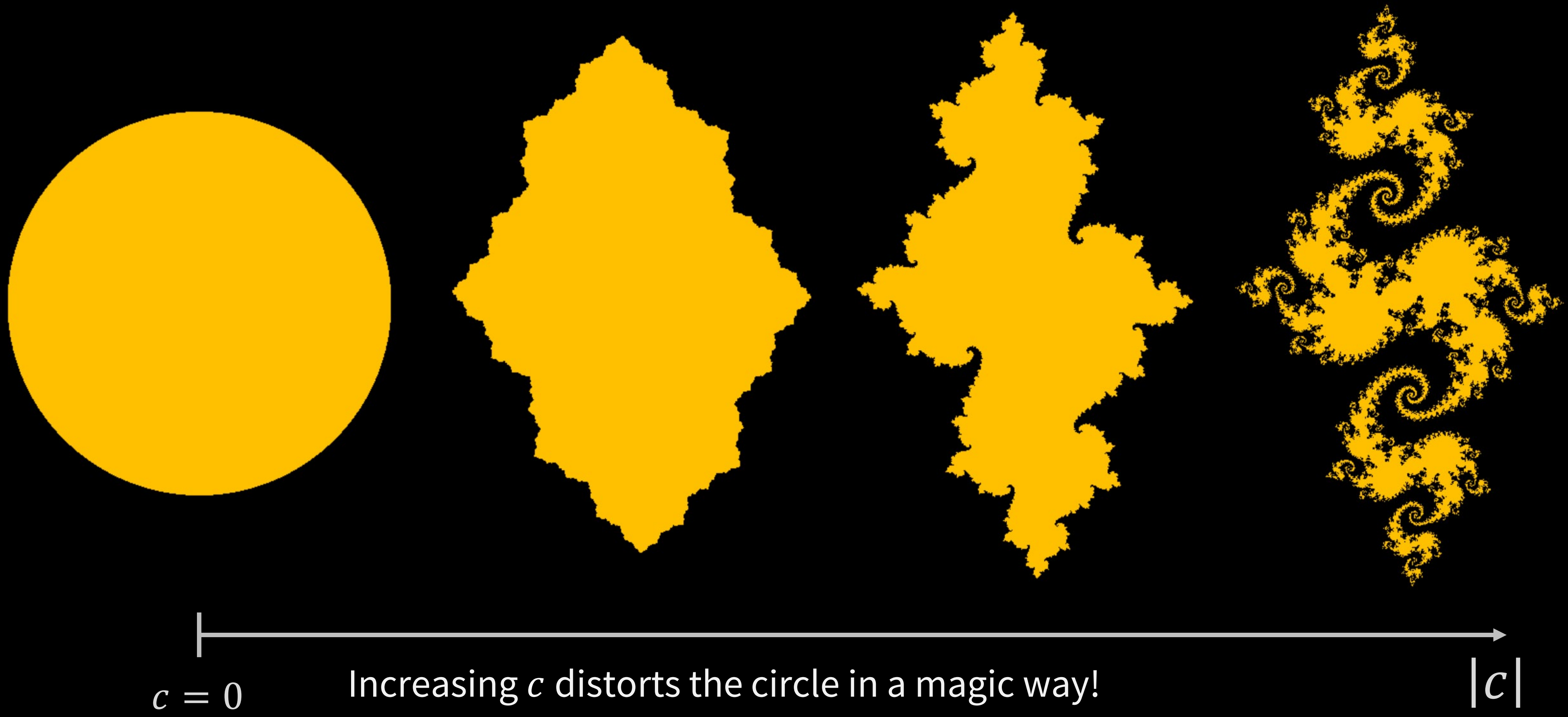
if at some point $r > 2$ we know it diverges

assume it does not diverge

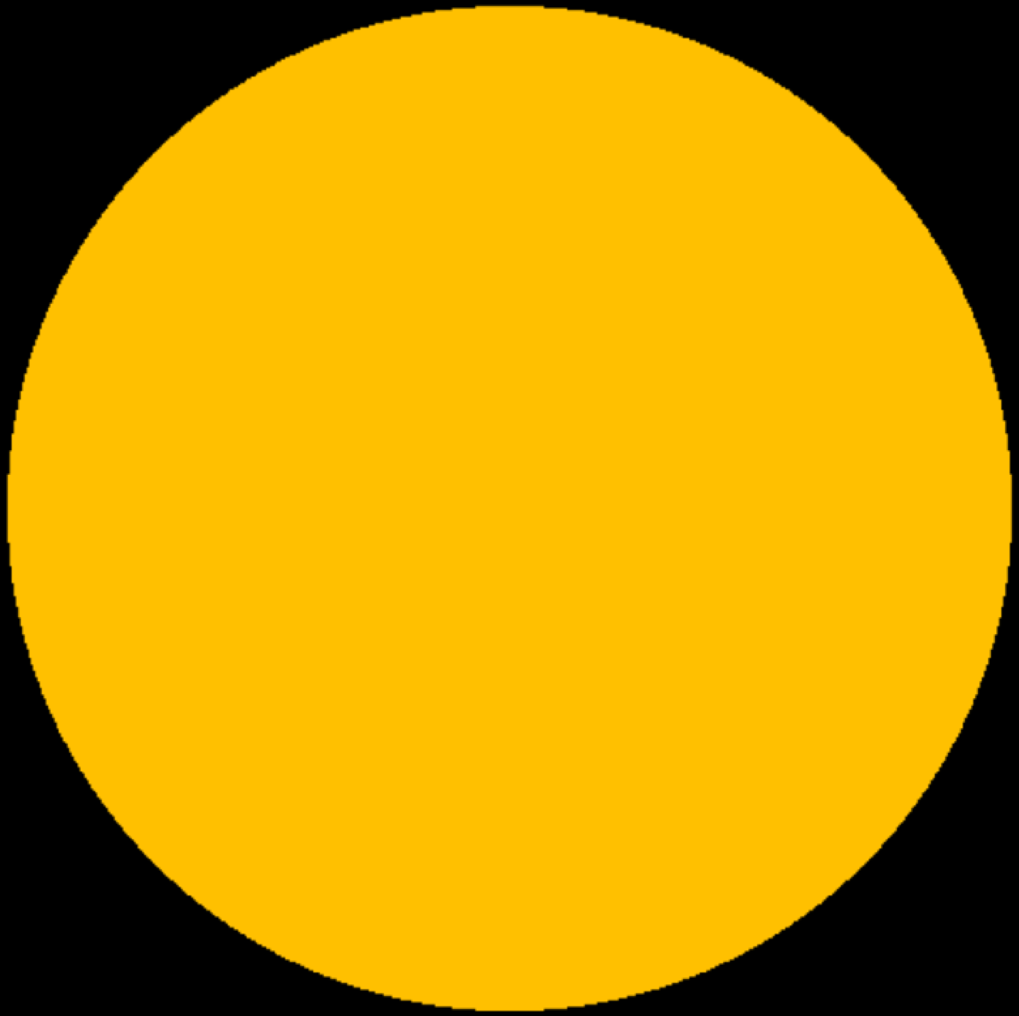
- Paint pixel yellow for non-divergent numbers
- The edge is the boundary

This is the most mind-boggling and magic program I have seen in my life!

# The Julia Set



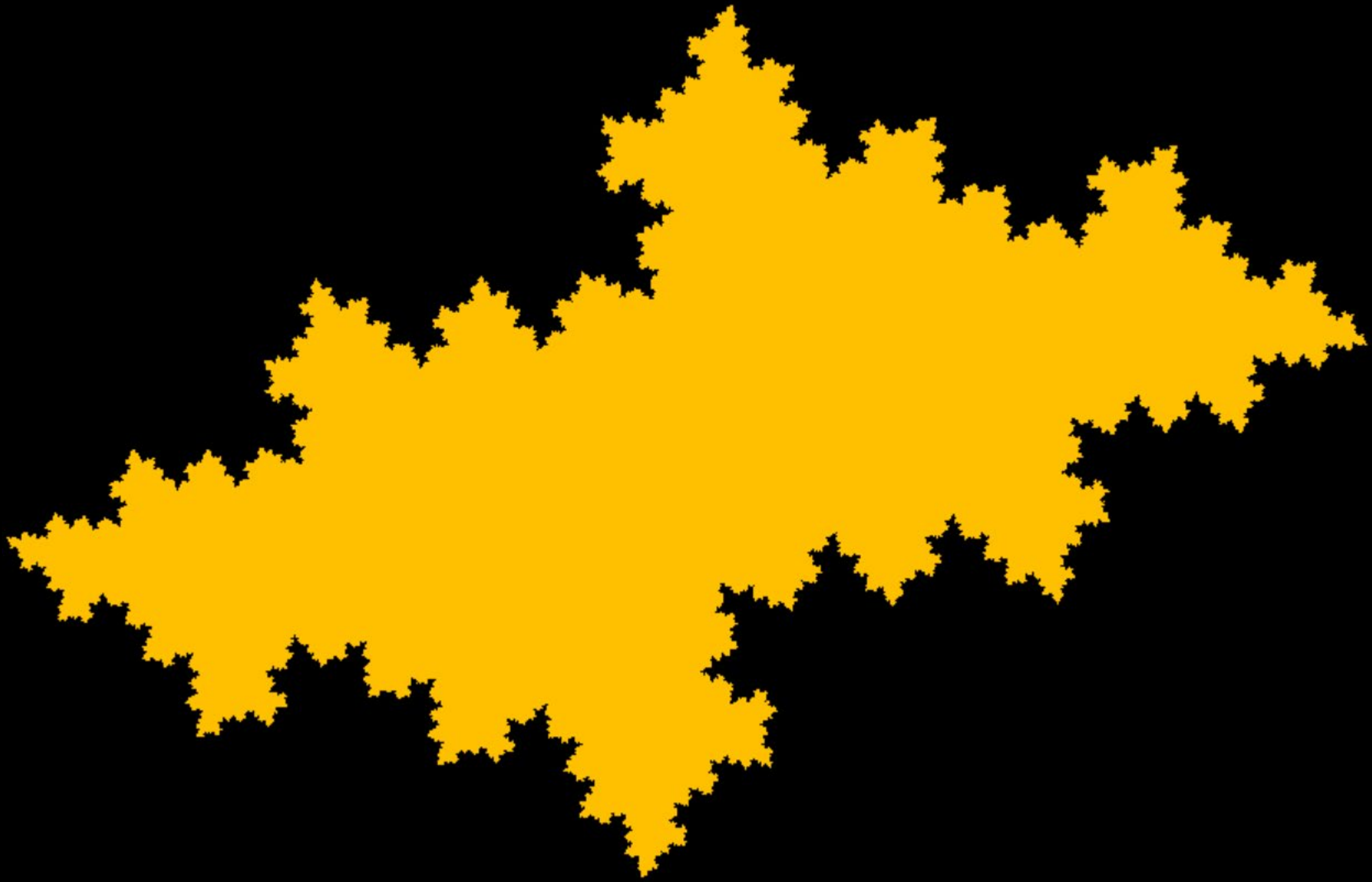
Increasing $c$ distorts the circle in a magic way!



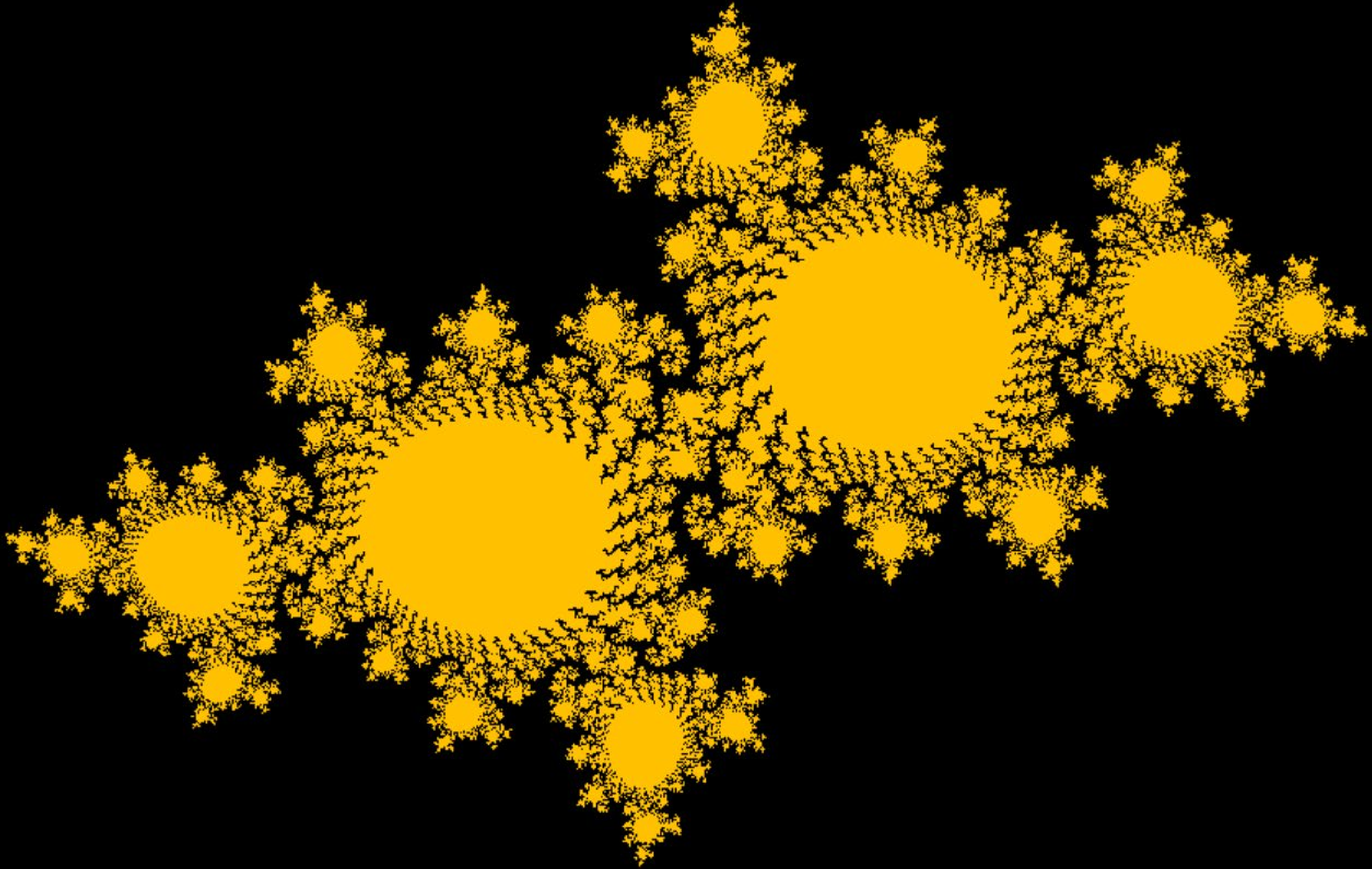
$$c = 0 + 0\,i$$

$$c = -0.18774 + 0.12075\, i$$

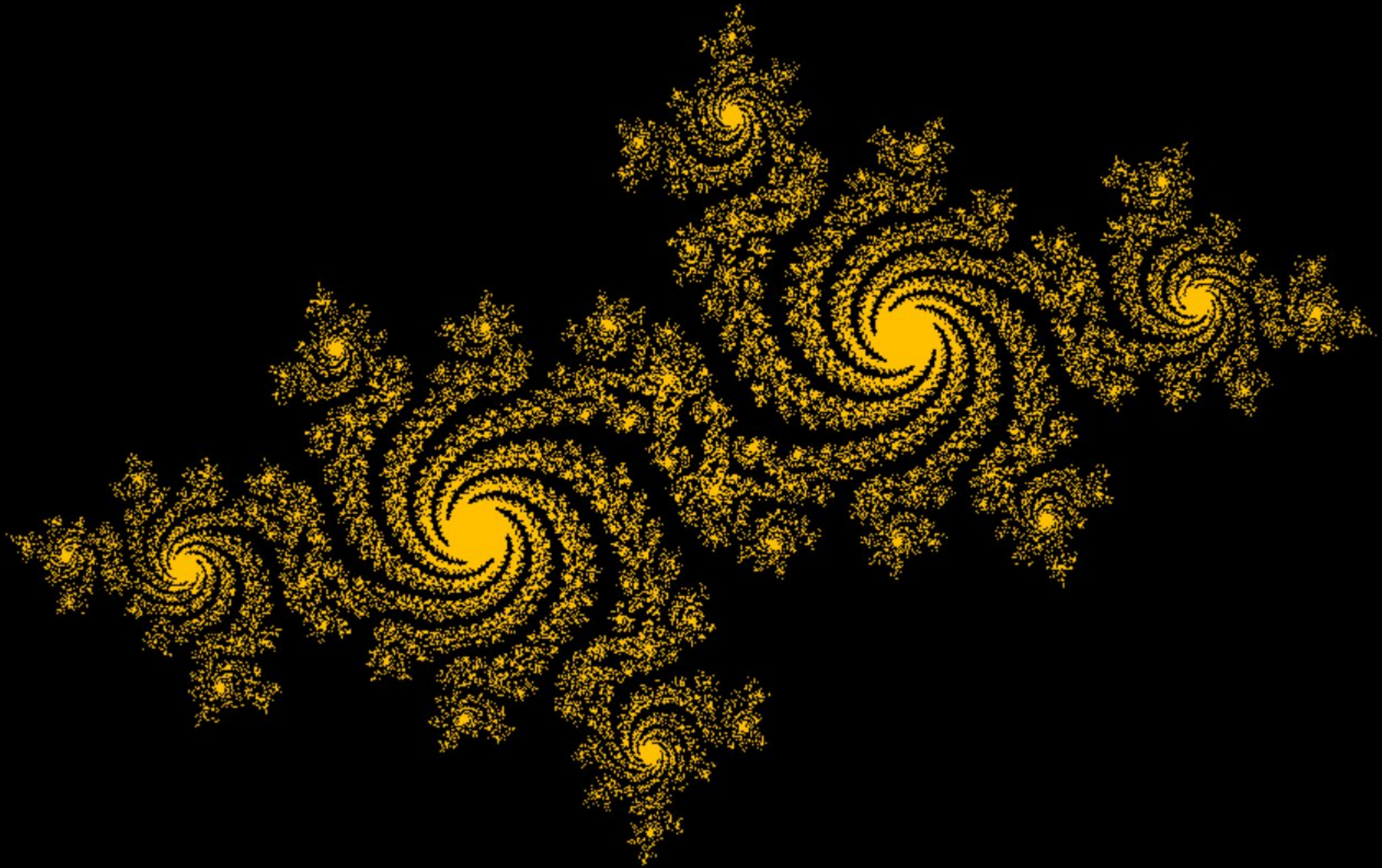
$$c = -0.50064 + 0.322\,i$$



$$c = -0.56322 + 0.36225\,i$$

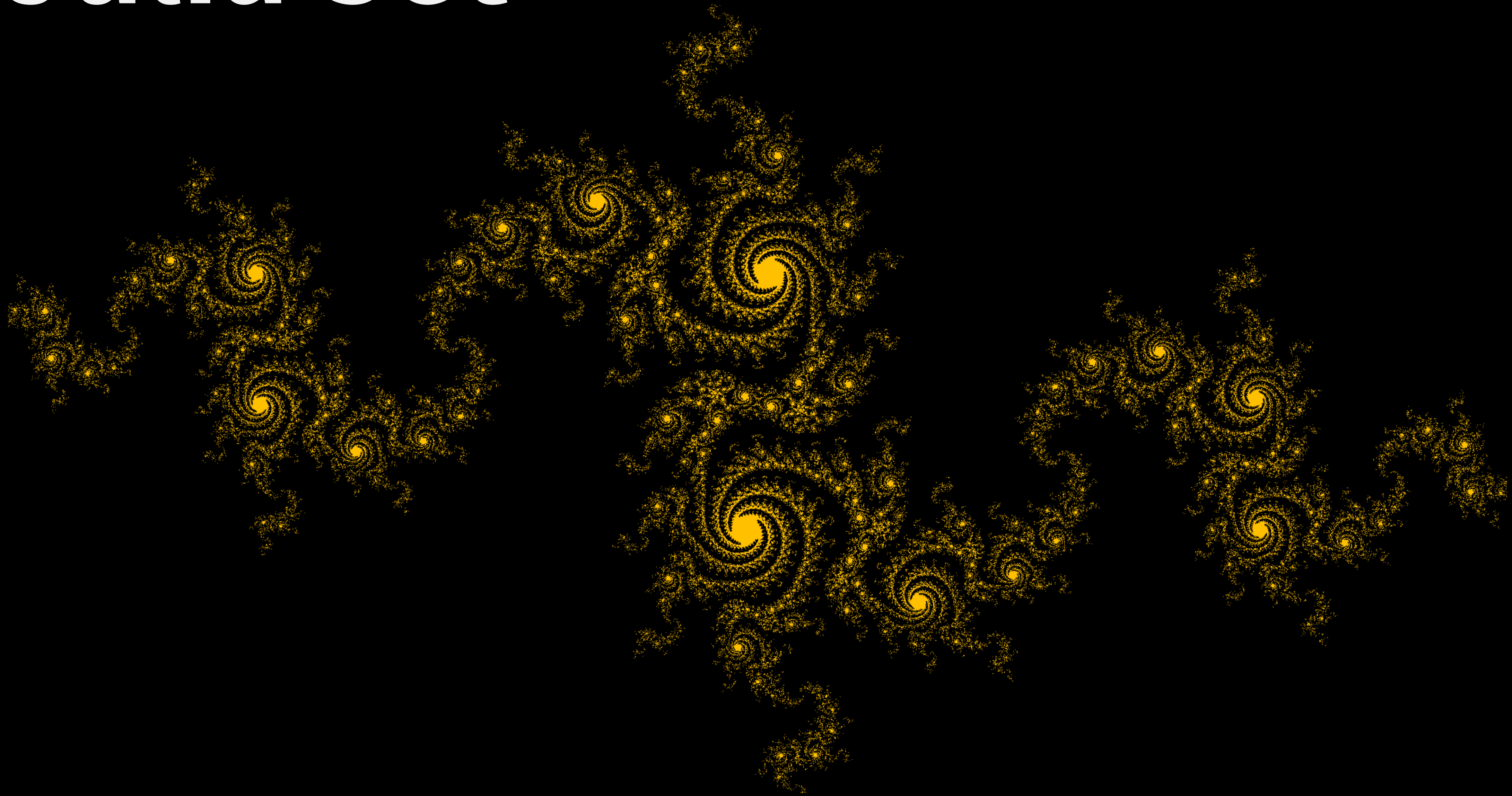


$$c = -0.619542 + 0.398475\, i$$

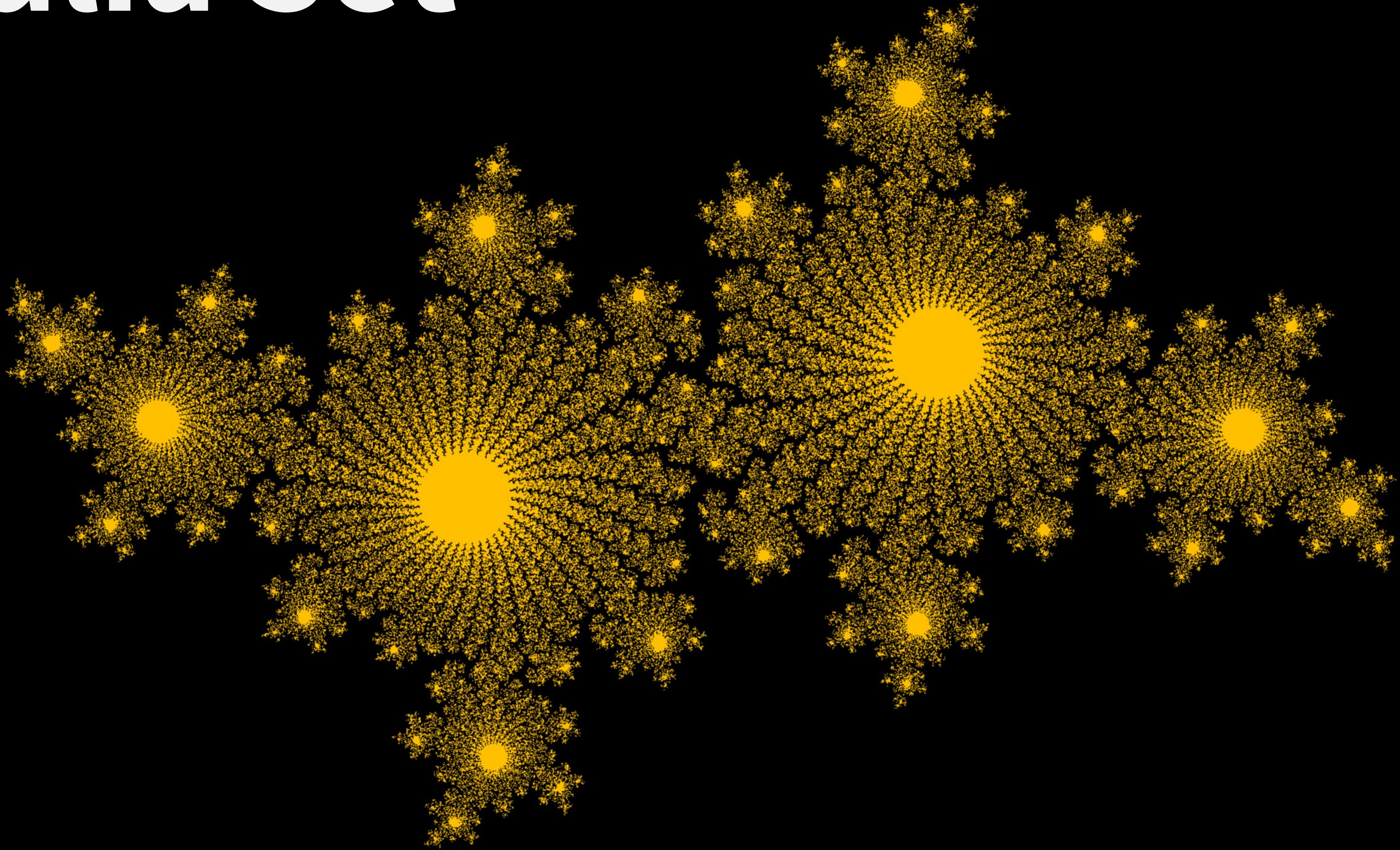


$$c = -0.6258 + 0.4025\, i$$

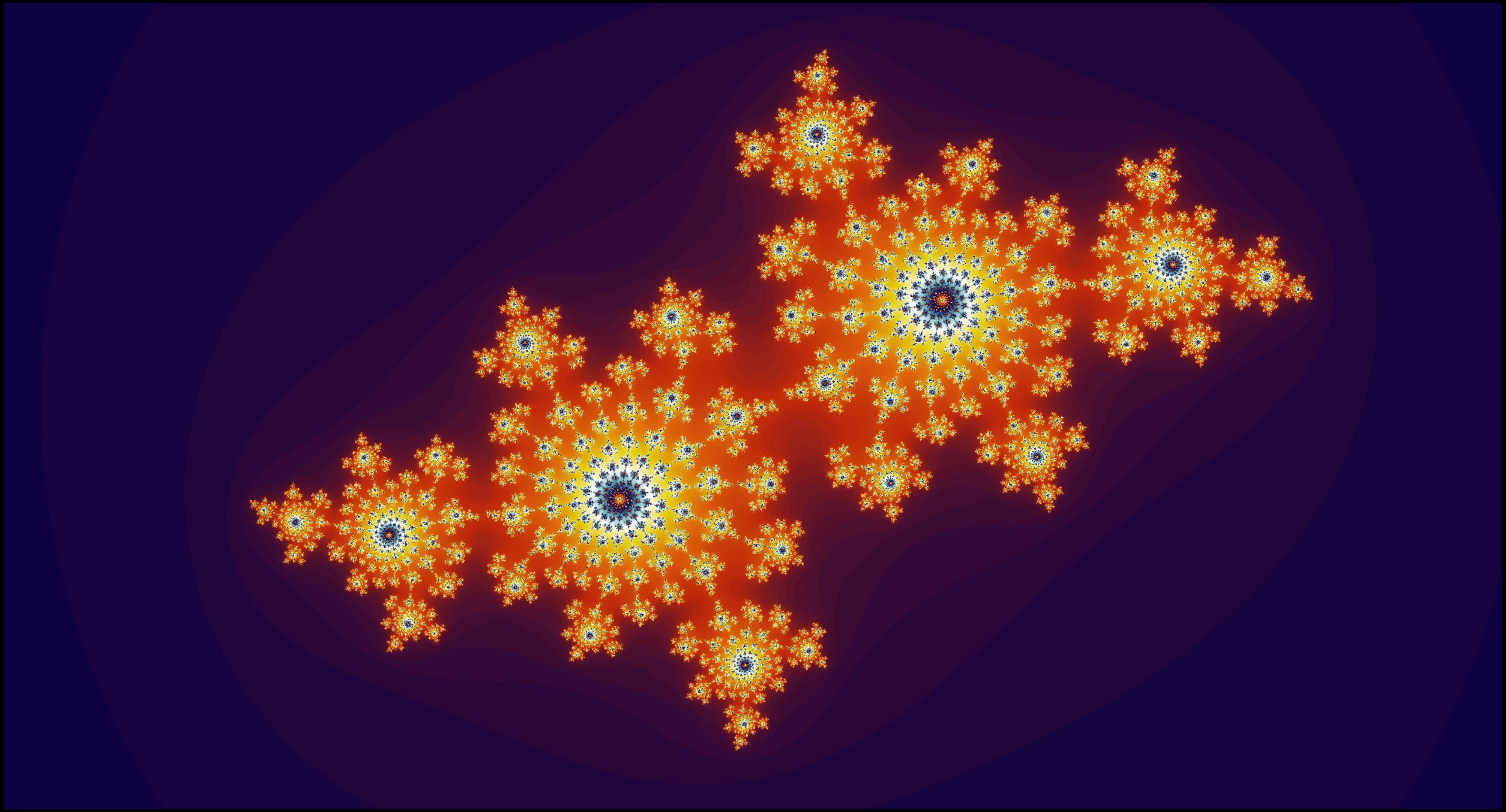
# Julia Set



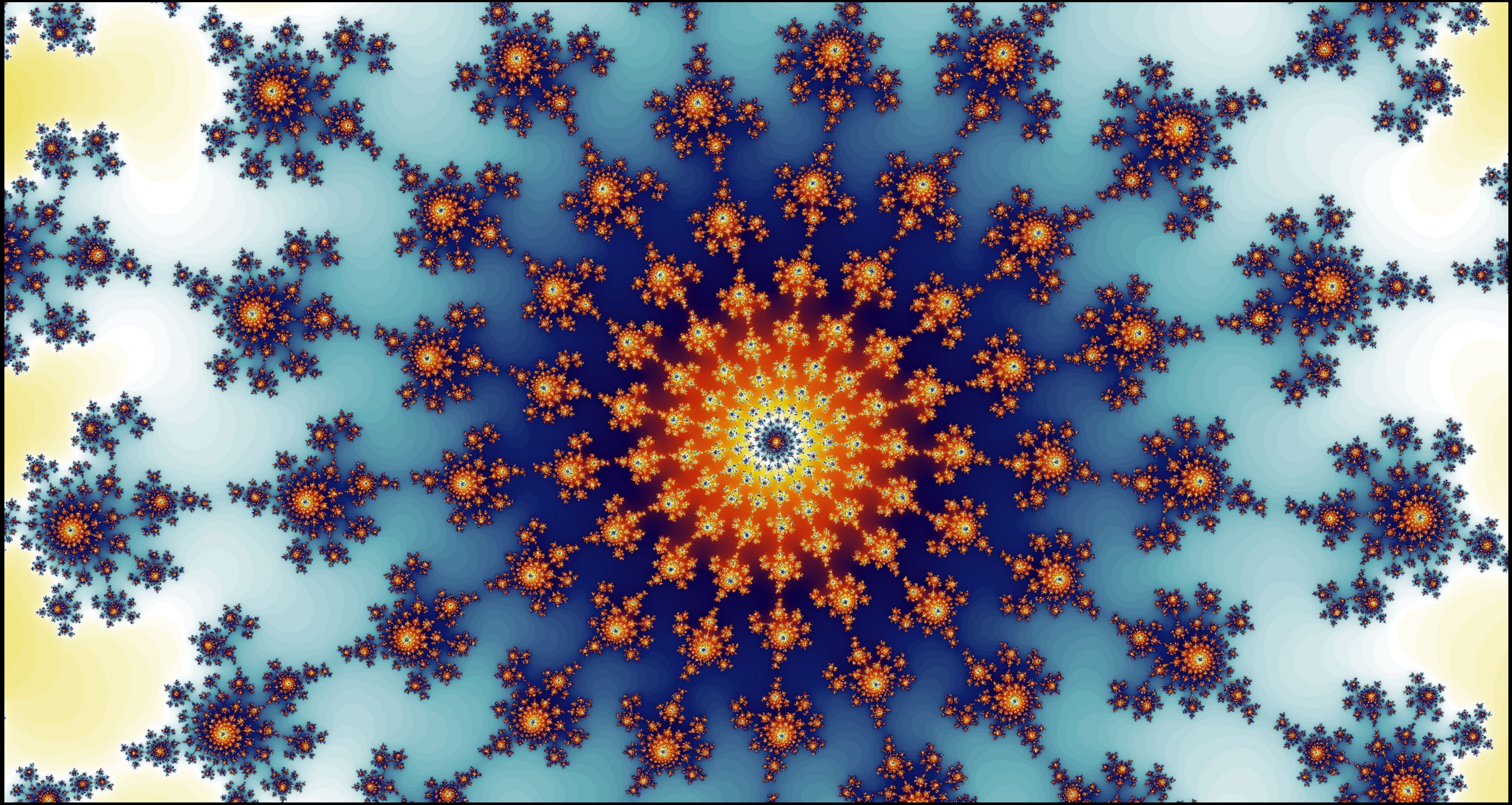
# Julia Set



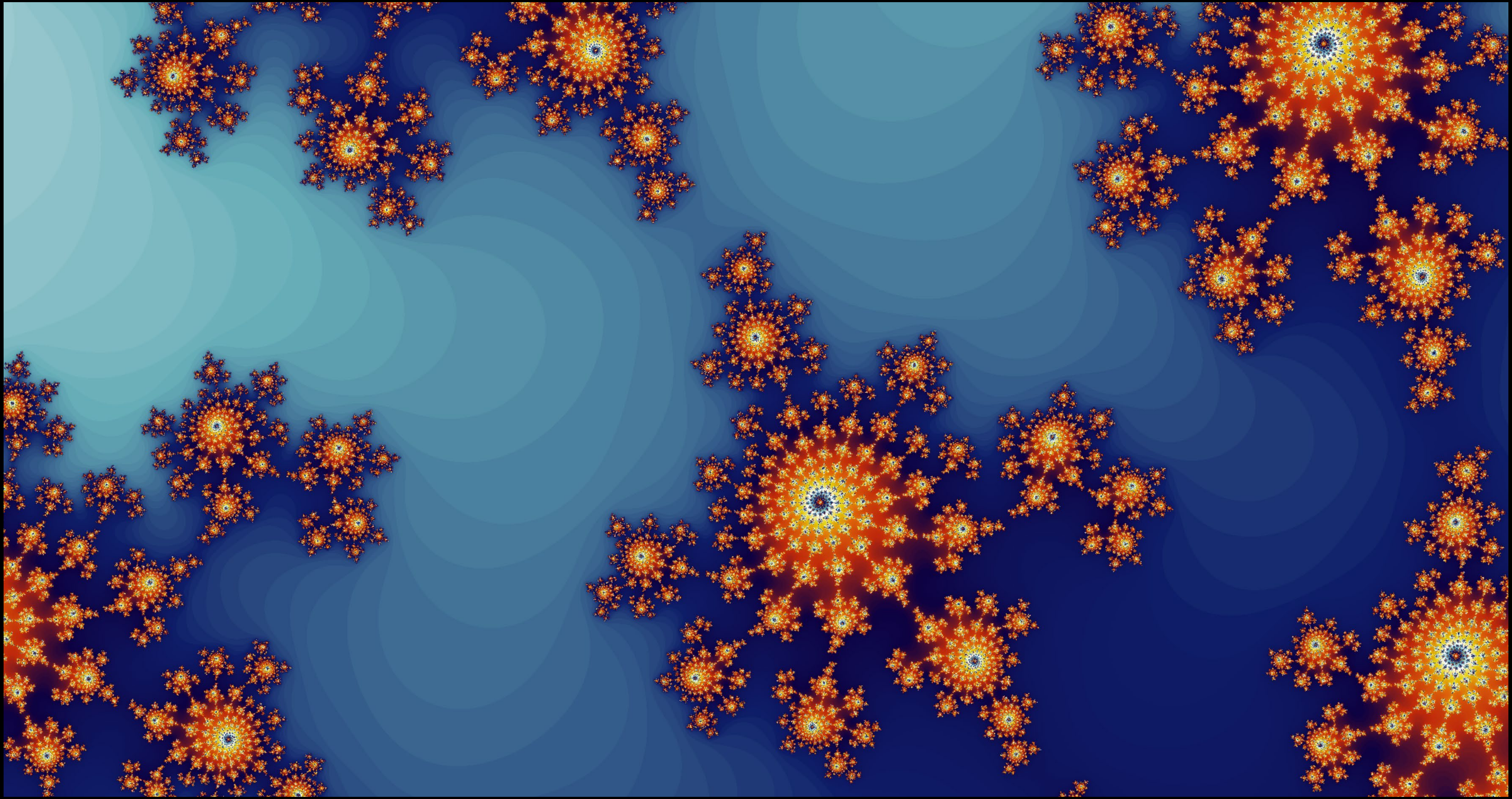
# Iteration Count as Color



# Zoom 1

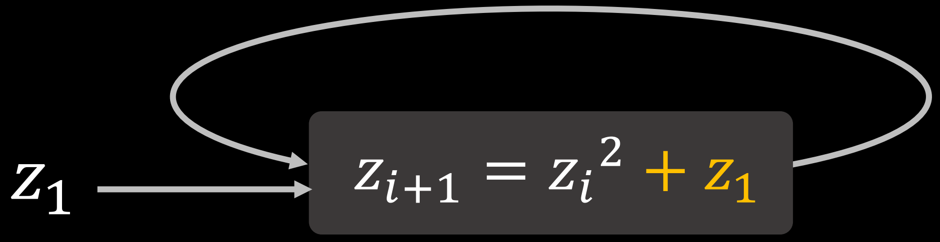


# Zoom 2



# The Mandelbrot Set

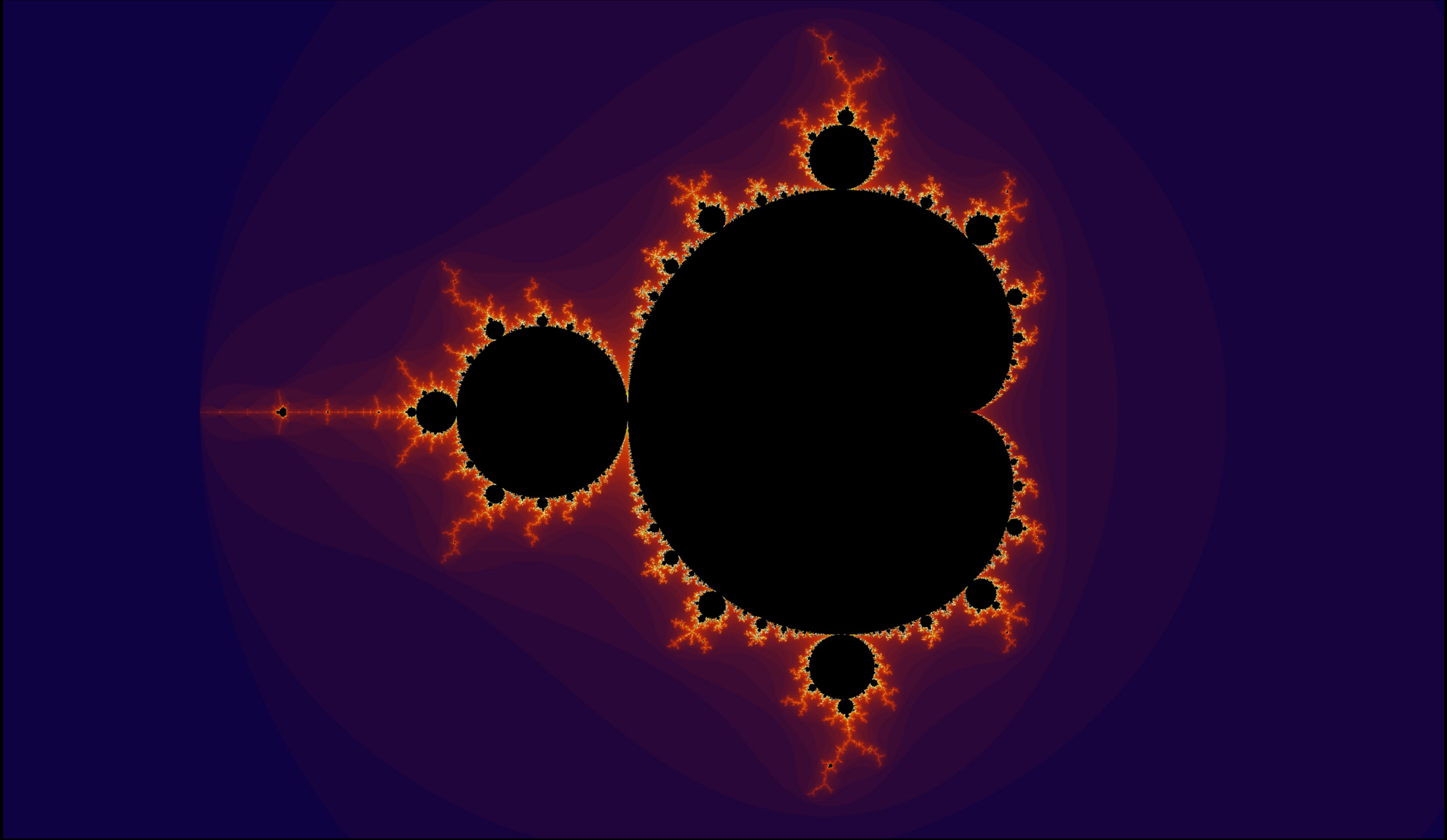
- A small modification of the game:



$z_1$

$$z_{i+1} = {z_i}^2 + z_1$$

- Use the starting number $z_1$ as the offset $c$
- No global parameter $c$



# The End of the Tutorial

## The Beginning of the Magic of Mathematics!